tion and interpolation arise in a variety of important practical applications of signals and systems, including communication systems, digital audio, high-definition television, and many other applications.

## Chapter 7 Problems

The first section of problems belongs to the basic category, and the answers are provided in the back of the book. The remaining two sections contain problems belonging to the basic and advanced categories, respectively.

### BASIC PROBLEMS WITH ANSWERS

**7.1.** A real-valued signal $x(t)$ is known to be uniquely determined by its samples when the sampling frequency is $\omega_s = 10{,}000\pi$. For what values of $\omega$ is $X(j\omega)$ guaranteed to be zero?

**7.2.** A continuous-time signal $x(t)$ is obtained at the output of an ideal lowpass filter with cutoff frequency $\omega_c = 1{,}000\pi$. If impulse-train sampling is performed on $x(t)$, which of the following sampling periods would guarantee that $x(t)$ can be recovered from its sampled version using an appropriate lowpass filter?

**(a)** $T = 0.5 \times 10^{-3}$
**(b)** $T = 2 \times 10^{-3}$
**(c)** $T = 10^{-4}$

**7.3.** The frequency which, under the sampling theorem, must be exceeded by the sampling frequency is called the *Nyquist rate*. Determine the Nyquist rate corresponding to each of the following signals:

**(a)** $x(t) = 1 + \cos(2{,}000\pi t) + \sin(4{,}000\pi t)$
**(b)** $x(t) = \frac{\sin(4{,}000\pi t)}{\pi t}$
**(c)** $x(t) = \left(\frac{\sin(4{,}000\pi t)}{\pi t}\right)^2$

**7.4.** Let $x(t)$ be a signal with Nyquist rate $\omega_0$. Determine the Nyquist rate for each of the following signals:

**(a)** $x(t) + x(t-1)$
**(b)** $\frac{dx(t)}{dt}$
**(c)** $x^2(t)$
**(d)** $x(t)\cos\omega_0 t$

**7.5.** Let $x(t)$ be a signal with Nyquist rate $\omega_0$. Also, let

$$y(t) = x(t)p(t-1),$$

where

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT), \text{ and } T < \frac{2\pi}{\omega_0}.$$

Specify the constraints on the magnitude and phase of the frequency response of a filter that gives $x(t)$ as its output when $y(t)$ is the input.

**7.6.** In the system shown in Figure P7.6, two functions of time, $x_1(t)$ and $x_2(t)$, are multiplied together, and the product $w(t)$ is sampled by a periodic impulse train. $x_1(t)$ is band limited to $\omega_1$, and $x_2(t)$ is band limited to $\omega_2$; that is,

$$X_1(j\omega) = 0, |\omega| \geq \omega_1,$$

$$X_2(j\omega) = 0, |\omega| \geq \omega_2.$$

Determine the *maximum* sampling interval $T$ such that $w(t)$ is recoverable from $w_p(t)$ through the use of an ideal lowpass filter.

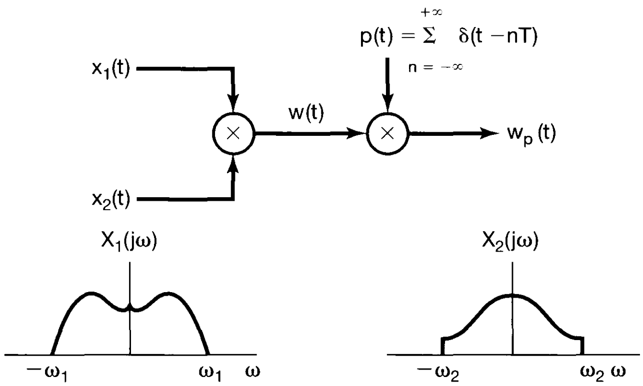



**Figure P7.6**

**7.7.** A signal $x(t)$ undergoes a zero-order hold operation with an effective sampling period $T$ to produce a signal $x_0(t)$. Let $x_1(t)$ denote the result of a first-order hold operation on the samples of $x(t)$; i.e.,

$$x_1(t) = \sum_{n=-\infty}^{\infty} x(nT)h_1(t - nT),$$

where $h_1(t)$ is the function shown in Figure P7.7. Specify the frequency response of a filter that produces $x_1(t)$ as its output when $x_0(t)$ is the input.

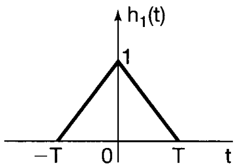


**Figure P7.7**

**7.8.** Consider a real, odd, and periodic signal $x(t)$ whose Fourier series representation may be expressed as

$$x(t) = \sum_{k=0}^{5} \left(\frac{1}{2}\right)^k \sin(k\pi t).$$

Let $\hat{x}(t)$ represent the signal obtained by performing impulse-train sampling on $x(t)$ using a sampling period of $T = 0.2$.

**(a)** Does aliasing occur when this impulse-train sampling is performed on $x(t)$?

**(b)** If $\hat{x}(t)$ is passed through an ideal lowpass filter with cutoff frequency $\pi/T$ and passband gain $T$, determine the Fourier series representation of the output signal $g(t)$.

**7.9.** Consider the signal

$$x(t) = \left(\frac{\sin 50\pi t}{\pi t}\right)^2,$$

which we wish to sample with a sampling frequency of $\omega_s = 150\pi$ to obtain a signal $g(t)$ with Fourier transform $G(j\omega)$. Determine the maximum value of $\omega_0$ for which it is guaranteed that

$$G(j\omega) = 75X(j\omega) \text{ for } |\omega| \leq \omega_0,$$

where $X(j\omega)$ is the Fourier transform of $x(t)$.

**7.10.** Determine whether each of the following statements is true or false:

**(a)** The signal $x(t) = u(t + T_0) - u(t - T_0)$ can undergo impulse-train sampling without aliasing, provided that the sampling period $T < 2T_0$.

**(b)** The signal $x(t)$ with Fourier transform $X(j\omega) = u(\omega + \omega_0) - u(\omega - \omega_0)$ can undergo impulse-train sampling without aliasing, provided that the sampling period $T < \pi/\omega_0$.

**(c)** The signal $x(t)$ with Fourier transform $X(j\omega) = u(\omega) - u(\omega - \omega_0)$ can undergo impulse-train sampling without aliasing, provided that the sampling period $T < 2\pi/\omega_0$.

**7.11.** Let $x_c(t)$ be a continuous-time signal whose Fourier transform has the property that $X_c(j\omega) = 0$ for $|\omega| \geq 2{,}000\pi$. A discrete-time signal

$$x_d[n] = x_c(n(0.5 \times 10^{-3}))$$

is obtained. For each of the following constraints on the Fourier transform $X_d(e^{j\omega})$ of $x_d[n]$, determine the corresponding constraint on $X_c(j\omega)$:

**(a)** $X_d(e^{j\omega})$ is real.

**(b)** The maximum value of $X_d(e^{j\omega})$ over all $\omega$ is 1.

**(c)** $X_d(e^{j\omega}) = 0$ for $\frac{3\pi}{4} \le |\omega| \le \pi$.

**(d)** $X_d(e^{j\omega}) = X_d(e^{j(\omega-\pi)})$.

**7.12.** A discrete-time signal $x_d[n]$ has a Fourier transform $X_d(e^{j\omega})$ with the property that $X_d(e^{j\omega}) = 0$ for $3\pi/4 \le |\omega| \le \pi$. The signal is converted into a continuous-time signal

$$x_c(t) = T \sum_{n=-\infty}^{\infty} x_d[n] \frac{\sin \frac{\pi}{T}(t - nT)}{\pi(t - nT)},$$

where $T = 10^{-3}$. Determine the values of $\omega$ for which the Fourier transform $X_c(j\omega)$ of $x_c(t)$ is guaranteed to be zero.

**7.13.** With reference to the filtering approach illustrated in Figure 7.24, assume that the sampling period used is $T$ and the input $x_c(t)$ is band limited, so that $X_c(j\omega) = 0$ for $|\omega| \ge \pi/T$. If the overall system has the property that $y_c(t) = x_c(t-2T)$, determine the impulse response $h[n]$ of the discrete-time filter in Figure 7.24.

**7.14.** Repeat the previous problem, except this time assume that

$$y_c(t) = \frac{d}{dt} x_c\left(t - \frac{T}{2}\right).$$

**7.15.** Impulse-train sampling of $x[n]$ is used to obtain

$$g[n] = \sum_{k=-\infty}^{\infty} x[n]\delta[n - kN].$$

If $X(e^{j\omega}) = 0$ for $3\pi/7 \le |\omega| \le \pi$, determine the largest value for the sampling interval $N$ which ensures that no aliasing takes place while sampling $x[n]$.

**7.16.** The following facts are given about the signal $x[n]$ and its Fourier transform:

1. $x[n]$ is real.
2. $X(e^{j\omega}) \ne 0$ for $0 < \omega < \pi$.
3. $x[n]\sum_{k=-\infty}^{\infty}\delta[n - 2k] = \delta[n]$.

Determine $x[n]$. You may find it useful to note that the signal $(\sin \frac{\pi}{2}n)/(\pi n)$ satisfies two of these conditions.

**7.17.** Consider an ideal discrete-time bandstop filter with impulse response $h[n]$ for which the frequency response in the interval $-\pi \leq \omega \leq \pi$ is

$$H(e^{j\omega}) = \begin{cases} 1, & |\omega| \leq \frac{\pi}{4} \text{ and } |\omega| \geq \frac{3\pi}{4} \\ 0, & \text{elsewhere} \end{cases}.$$

Determine the frequency response of the filter whose impulse response is $h[2n]$.

**7.18.** Suppose the impulse response of an ideal discrete-time lowpass filter with cutoff frequency $\pi/2$ is interpolated (in accordance with Figure 7.37) to obtain an upsampling by a factor of 2. What is the frequency response corresponding to this upsampled impulse response?

**7.19.** Consider the system shown in Figure P7.19, with input $x[n]$ and the corresponding output $y[n]$. The zero-insertion system inserts two points with zero amplitude between each of the sequence values in $x[n]$. The decimation is defined by

$$y[n] = w[5n],$$

where $w[n]$ is the input sequence for the decimation system. If the input is of the form

$$x[n] = \frac{\sin \omega_1 n}{\pi n},$$

determine the output $y[n]$ for the following values of $\omega_1$:

**(a)** $\omega_1 \leq \frac{3\pi}{5}$

**(b)** $\omega_1 > \frac{3\pi}{5}$

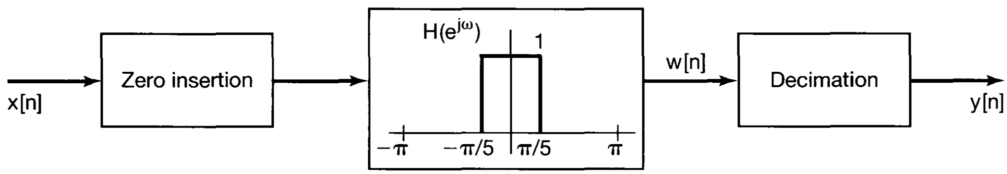
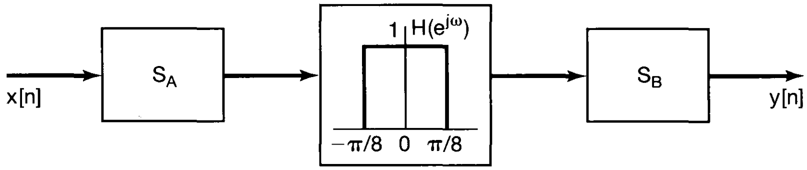



**Figure P7.19**

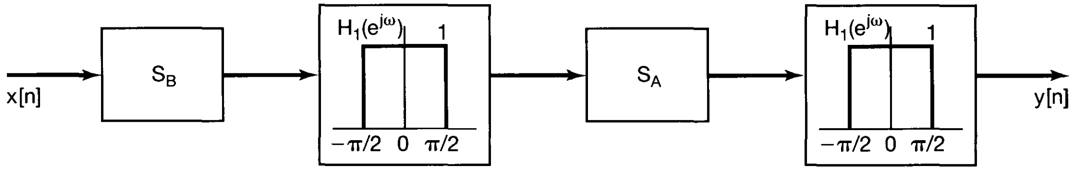
**7.20.** Two discrete-time systems $S_1$ and $S_2$ are proposed for implementing an ideal lowpass filter with cutoff frequency $\pi/4$. System $S_1$ is depicted in Figure P7.20(a). System $S_2$ is depicted in Figure P7.20(b). In these figures, $S_A$ corresponds to a zero-insertion system that inserts one zero after every input sample, while $S_B$ corresponds to a decimation system that extracts every second sample of its input.

**(a)** Does the proposed system $S_1$ correspond to the desired ideal lowpass filter?

**(b)** Does the proposed system $S_2$ correspond to the desired ideal lowpass filter?




(a)




(b)

**Figure P7.20**

## BASIC PROBLEMS

**7.21.** A signal $x(t)$ with Fourier transform $X(j\omega)$ undergoes impulse-train sampling to generate

$$x_p(t) = \sum_{n=-\infty}^{\infty} x(nT)\,\delta(t - nT)$$

where $T = 10^{-4}$. For each of the following sets of constraints on $x(t)$ and/or $X(j\omega)$, does the sampling theorem (see Section 7.1.1) guarantee that $x(t)$ can be recovered exactly from $x_p(t)$?

**(a)** $X(j\omega) = 0$ for $|\omega| > 5000\pi$
**(b)** $X(j\omega) = 0$ for $|\omega| > 15000\pi$
**(c)** $\mathcal{R}e\{X(j\omega)\} = 0$ for $|\omega| > 5000\pi$
**(d)** $x(t)$ real and $X(j\omega) = 0$ for $\omega > 5000\pi$
**(e)** $x(t)$ real and $X(j\omega) = 0$ for $\omega < -15000\pi$
**(f)** $X(j\omega) * X(j\omega) = 0$ for $|\omega| > 15000\pi$
**(g)** $|X(j\omega)| = 0$ for $\omega > 5000\pi$

**7.22.** The signal $y(t)$ is generated by convolving a band-limited signal $x_1(t)$ with another band-limited signal $x_2(t)$, that is,

$$y(t) = x_1(t) * x_2(t)$$

where

$$X_1(j\omega) = 0 \qquad \text{for } |\omega| > 1000\pi$$
$$X_2(j\omega) = 0 \qquad \text{for } |\omega| > 2000\pi.$$

Impulse-train sampling is performed on $y(t)$ to obtain

$$y_p(t) = \sum_{n=-\infty}^{+\infty} y(nT)\delta(t - nT).$$

Specify the range of values for the sampling period $T$ which ensures that $y(t)$ is recoverable from $y_p(t)$.

**7.23.** Shown in Figure P7.23 is a system in which the sampling signal is an impulse train with alternating sign. The Fourier transform of the input signal is as indicated in the figure.

**(a)** For $\Delta < \pi/(2\omega_M)$, sketch the Fourier transform of $x_p(t)$ and $y(t)$.

**(b)** For $\Delta < \pi/(2\omega_M)$, determine a system that will recover $x(t)$ from $x_p(t)$.

**(c)** For $\Delta < \pi/(2\omega_M)$, determine a system that will recover $x(t)$ from $y(t)$.

**(d)** What is the *maximum* value of $\Delta$ in relation to $\omega_M$ for which $x(t)$ can be recovered from either $x_p(t)$ or $y(t)$?

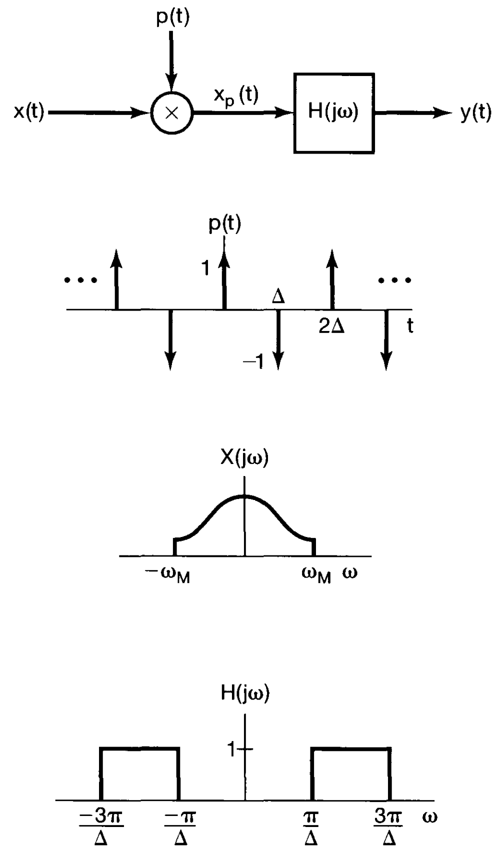



**Figure P7.23**

**7.24.** Shown in Figure P7.24 is a system in which the input signal is multiplied by a periodic square wave. The period of $s(t)$ is $T$. The input signal is band limited with $|X(j\omega)| = 0$ for $|\omega| \geq \omega_M$.

**(a)** For $\Delta = T/3$, determine, in terms of $\omega_M$, the maximum value of $T$ for which there is no aliasing among the replicas of $X(j\omega)$ in $W(j\omega)$.

**(b)** For $\Delta = T/4$, determine, in terms of $\omega_M$, the maximum value of $T$ for which there is no aliasing among the replicas of $X(j\omega)$ in $W(j\omega)$.

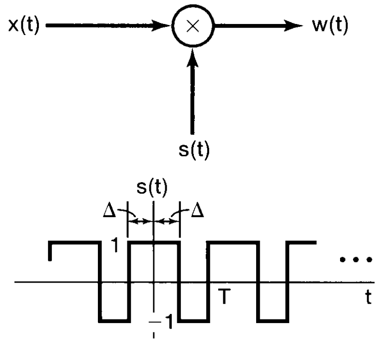



**Figure P7.24**

**7.25.** In Figure P7.25 is a sampler, followed by an ideal lowpass filter, for reconstruction of $x(t)$ from its samples $x_p(t)$. From the sampling theorem, we know that if $\omega_s = 2\pi/T$ is greater than twice the highest frequency present in $x(t)$ and $\omega_c = \omega_s/2$, then the reconstructed signal $x_r(t)$ will exactly equal $x(t)$. If this condition on the bandwidth of $x(t)$ is violated, then $x_r(t)$ will *not* equal $x(t)$. We seek to show in this problem that if $\omega_c = \omega_s/2$, then for any choice of $T$, $x_r(t)$ and $x(t)$ will always be equal at the sampling instants; that is,

$$x_r(kT) = x(kT),\ k = 0, \pm 1, \pm 2, \ldots.$$

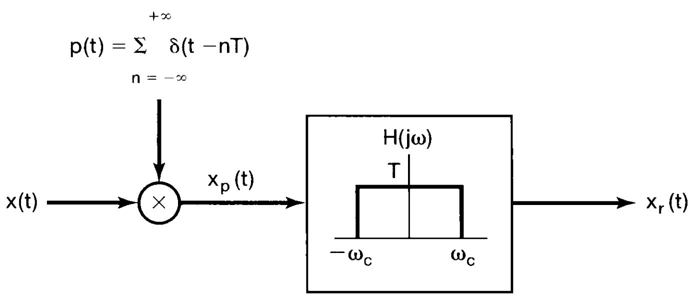



**Figure P7.25**

To obtain this result, consider eq. (7.11), which expresses $x_r(t)$ in terms of the samples of $x(t)$:

$$x_r(t) = \sum_{n=-\infty}^{\infty} x(nT)T\frac{\omega_c}{\pi}\frac{\sin[\omega_c(t - nT)]}{\omega_c(t - nT)}.$$

With $\omega_c = \omega_s/2$, this becomes

$$x_r(t) = \sum_{n=-\infty}^{\infty} x(nT)\frac{\sin\left[\frac{\pi}{T}(t - nT)\right]}{\frac{\pi}{T}(t - nT)}. \qquad \text{(P7.25–1)}$$

By considering the values of $\alpha$ for which $[\sin(\alpha)]/\alpha = 0$, show from eq. (P7.25–1) that, without any restrictions on $x(t)$, $x_r(kT) = x(kT)$ for any integer value of $k$.

**7.26.** The sampling theorem, as we have derived it, states that a signal $x(t)$ must be sampled at a rate greater than its bandwidth (or equivalently, a rate greater than twice its highest frequency). This implies that if $x(t)$ has a spectrum as indicated in Figure P7.26(a) then $x(t)$ must be sampled at a rate greater than $2\omega_2$. However, since the signal has most of its energy concentrated in a narrow band, it would seem reasonable to expect that a sampling rate lower than twice the highest frequency could be used. A signal whose energy is concentrated in a frequency band is often referred to as a *bandpass signal*. There are a variety of techniques for sampling such signals, generally referred to as *bandpass-sampling* techniques.

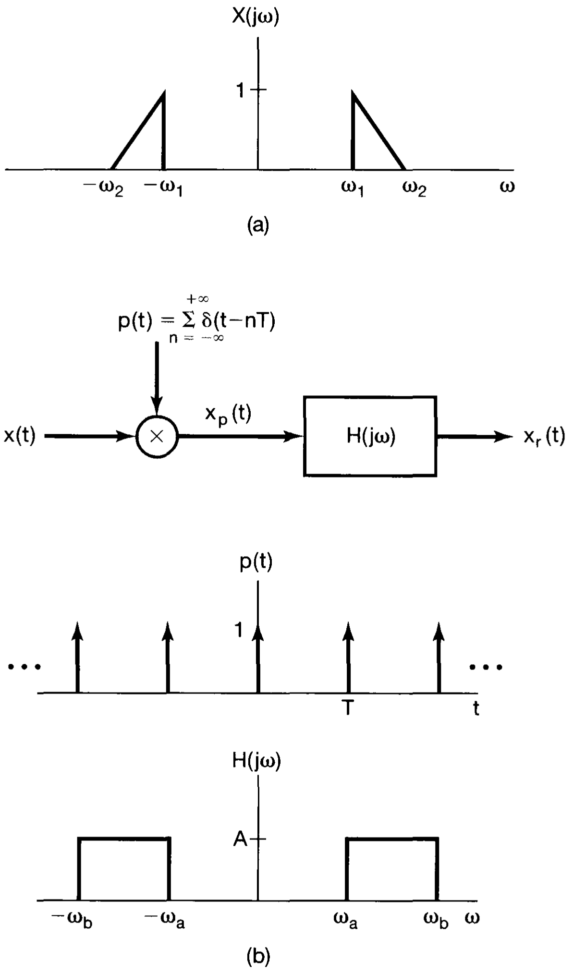



Figure P7.26

To examine the possibility of sampling a bandpass signal as a rate less than the total bandwidth, consider the system shown in Figure P7.26(b). Assuming that $\omega_1 > \omega_2 - \omega_1$, find the maximum value of $T$ and the values of the constants $A$, $\omega_a$, and $\omega_b$ such that $x_r(t) = x(t)$.

**7.27.** In Problem 7.26, we considered one procedure for bandpass sampling and reconstruction. Another procedure, used when $x(t)$ is real, consists of multiplying $x(t)$ by a complex-exponential and then sampling the product. The sampling system is shown in Figure P7.27(a). With $x(t)$ real and with $X(j\omega)$ nonzero only for $\omega_1 < |\omega| < \omega_2$, the frequency is chosen to be $\omega_0 = (1/2)(\omega_1 + \omega_2)$, and the lowpass filter $H_1(j\omega)$ has cutoff frequency $(1/2)(\omega_2 - \omega_1)$.

**(a)** For $X(j\omega)$ as shown in Figure P7.27(b), sketch $X_p(j\omega)$.

**(b)** Determine the maximum sampling period $T$ such that $x(t)$ is recoverable from $x_p(t)$.

**(c)** Determine a system to recover $x(t)$ from $x_p(t)$.

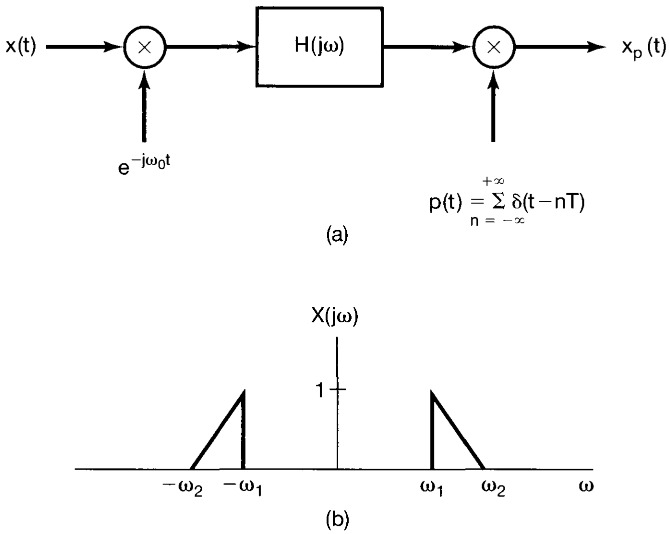



**Figure P7.27**

**7.28.** Figure P7.28(a) shows a system that converts a continuous-time signal to a discrete-time signal. The input $x(t)$ is periodic with a period of 0.1 second. The Fourier series coefficients of $x(t)$ are

$$a_k = \left(\frac{1}{2}\right)^{|k|}, \quad -\infty < k < +\infty.$$

The lowpass filter $H(j\omega)$ has the frequency response shown in Figure P7.28(b). The sampling period $T = 5 \times 10^{-3}$ second.

**(a)** Show that $x[n]$ is a periodic sequence, and determine its period.

**(b)** Determine the Fourier series coefficients of $x[n]$.

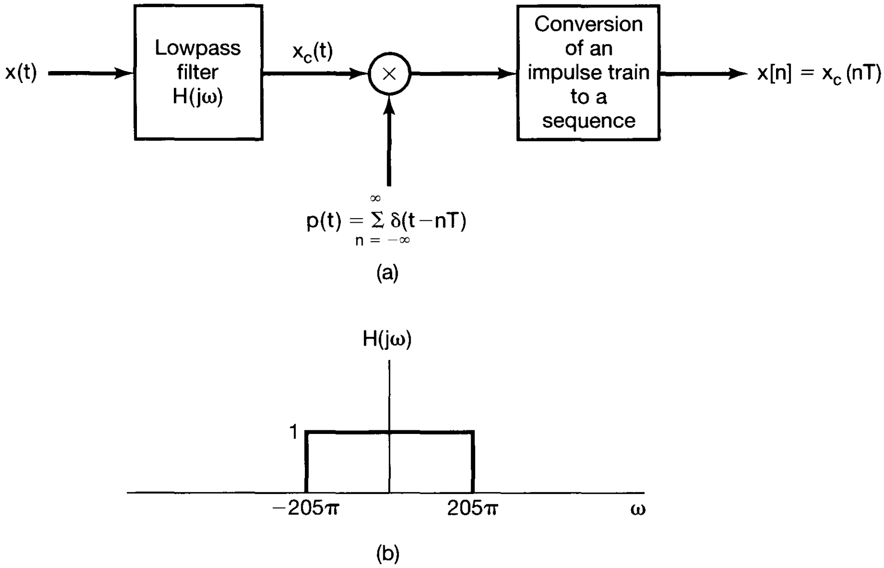



**Figure P7.28**

**7.29.** Figure P7.29(a) shows the overall system for filtering a continuous-time signal using a discrete-time filter. If $X_c(j\omega)$ and $H(e^{j\omega})$ are as shown in Figure P7.29(b), with $1/T = 20$ kHz, sketch $X_p(j\omega)$, $X(e^{j\omega})$, $Y(e^{j\omega})$, $Y_p(j\omega)$, and $Y_c(j\omega)$.

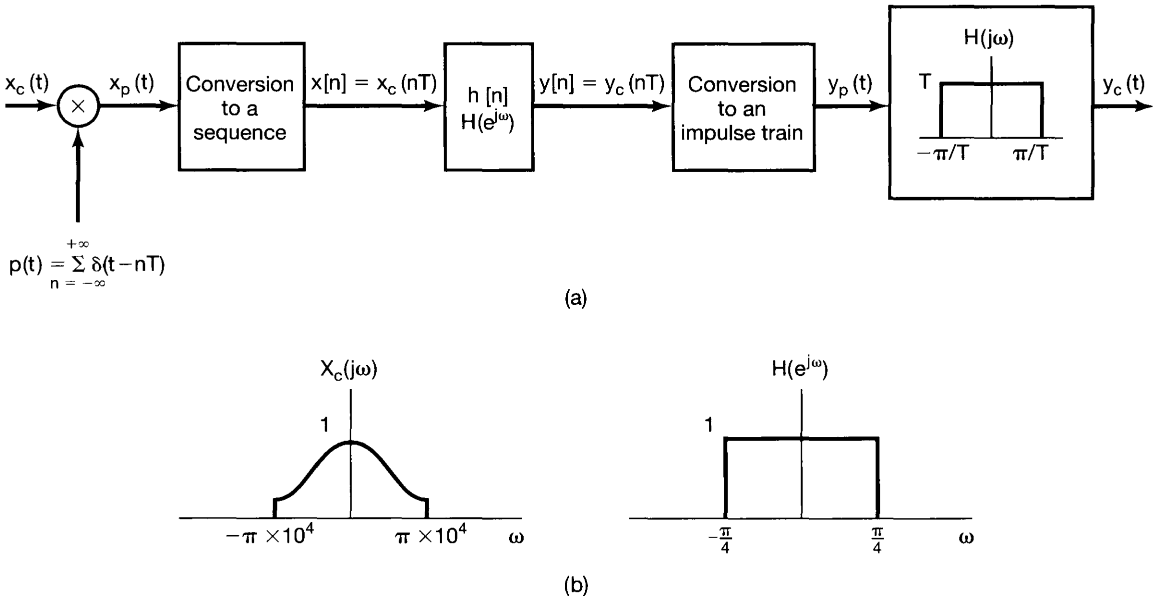



**Figure P7.29**

**7.30.** Figure P7.30 shows a system consisting of a continuous-time LTI system followed by a sampler, conversion to a sequence, and an LTI discrete-time system. The continuous-time LTI system is causal and satisfies the linear, constant-coefficient differential equation

$$\frac{dy_c(t)}{dt} + y_c(t) = x_c(t).$$

The input $x_c(t)$ is a unit impulse $\delta(t)$.

**(a)** Determine $y_c(t)$.

**(b)** Determine the frequency response $H(e^{j\omega})$ and the impulse response $h[n]$ such that $w[n] = \delta[n]$.

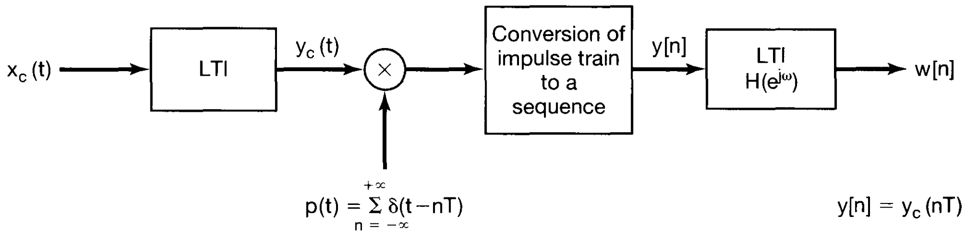



**Figure P7.30**

**7.31.** Shown in Figure P7.31 is a system that processes continuous-time signals using a digital filter $h[n]$ that is linear and causal with difference equation

$$y[n] = \frac{1}{2}y[n-1] + x[n].$$

For input signals that are band limited such that $X_c(j\omega) = 0$ for $|\omega| > \pi/T$, the system in the figure is equivalent to a continuous-time LTI system.

Determine the frequency response $H_c(j\omega)$ of the equivalent overall system with input $x_c(t)$ and output $y_c(t)$.

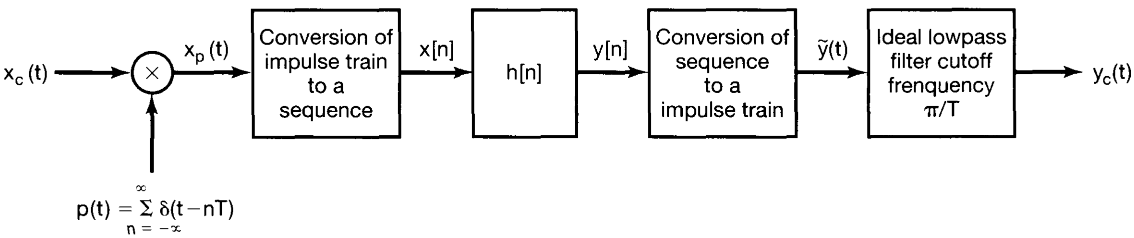



**Figure P7.31**

**7.32.** A signal $x[n]$ has a Fourier transform $X(e^{j\omega})$ that is zero for $(\pi/4) \le |\omega| \le \pi$. Another signal

$$g[n] = x[n] \sum_{k=-\infty}^{\infty} \delta[n-1-4k]$$

is generated. Specify the frequency response $H(e^{j\omega})$ of a lowpass filter that produces $x[n]$ as output when $g[n]$ is the input.

**7.33.** A signal $x[n]$ with Fourier transform $X(e^{j\omega})$ has the property that

$$\left(x[n] \sum_{k=-\infty}^{\infty} \delta[n-3k]\right) * \left(\frac{\sin \frac{\pi}{3} n}{\frac{\pi}{3} n}\right) = x[n].$$

For what values of $\omega$ is it guaranteed that $X(e^{j\omega}) = 0$?

**7.34.** A real-valued discrete-time signal $x[n]$ has a Fourier transform $X(e^{j\omega})$ that is zero for $3\pi/14 \leq |\omega| \leq \pi$. The nonzero portion of the Fourier transform of one period of $X(e^{j\omega})$ can be made to occupy the region $|\omega| < \pi$ by first performing upsampling by a factor of $L$ and then performing downsampling by a factor of $M$. Specify the values of $L$ and $M$.

**7.35.** Consider a discrete-time sequence $x[n]$ from which we form two new sequences, $x_p[n]$ and $x_d[n]$, where $x_p[n]$ corresponds to sampling $x[n]$ with a sampling period of 2 and $x_d[n]$ corresponds to decimating $x[n]$ by a factor of 2, so that

$$x_p[n] = \begin{cases} x[n], & n = 0, \pm 2, \pm 4, \ldots \\ 0, & n = \pm 1, \pm 3, \ldots \end{cases}$$

and

$$x_d[n] = x[2n].$$

**(a)** If $x[n]$ is as illustrated in Figure P7.35(a), sketch the sequences $x_p[n]$ and $x_d[n]$.
**(b)** If $X(e^{j\omega})$ is as shown in Figure P7.35(b), sketch $X_p(e^{j\omega})$ and $X_d(e^{j\omega})$.

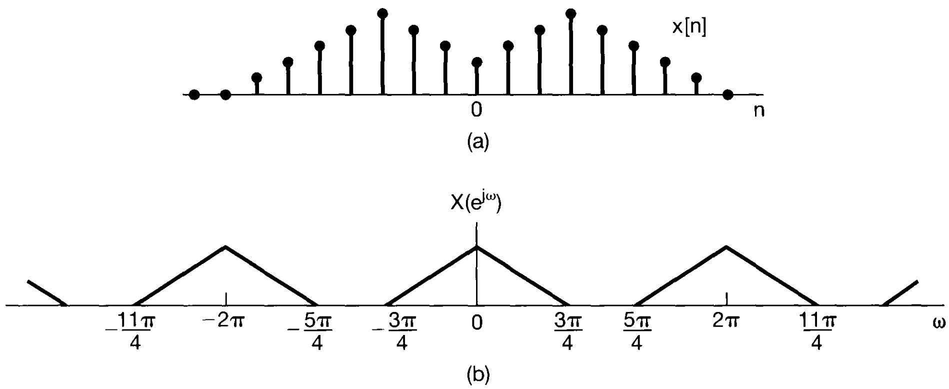



**Figure P7.35**

## ADVANCED PROBLEMS

**7.36** Let $x(t)$ be a band-limited signal such that $X(j\omega) = 0$ for $|\omega| \geq \frac{\pi}{T}$.
**(a)** If $x(t)$ is sampled using a sampling period $T$, determine an interpolating function

$g(t)$ such that

$$\frac{dx(t)}{dt} = \sum_{n=-\infty}^{\infty} x(nT)g(t - nT).$$

**(b)** Is the function $g(t)$ unique?

**7.37.** A signal limited in bandwidth to $|\omega| < W$ can be recovered from nonuniformly spaced samples as long as the average sample density is $2(W/2\pi)$ samples per second. This problem illustrates a particular example of nonuniform sampling. Assume that in Figure P7.37(a):

1. $x(t)$ is band limited; $X(j\omega) = 0, |\omega| > W$.
2. $p(t)$ is a nonuniformly spaced periodic pulse train, as shown in Figure P7.37(b).
3. $f(t)$ is a periodic waveform with period $T = 2\pi/W$. Since $f(t)$ multiplies an impulse train, only its values $f(0) = a$ and $f(\Delta) = b$ at $t = 0$ and $t = \Delta$, respectively, are significant.
4. $H_1(j\omega)$ is a 90° phase shifter; that is,

$$H_1(j\omega) = \begin{cases} j, & \omega > 0 \\ -j, & \omega < 0 \end{cases}.$$

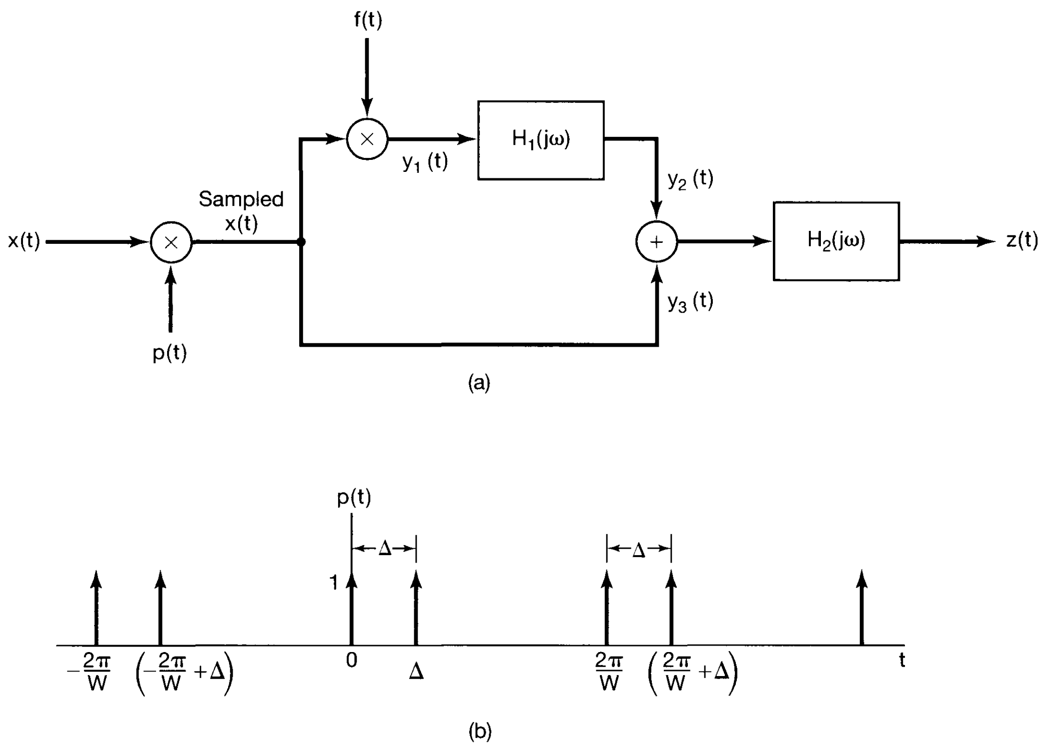



**Figure P7.37**

5. $H_2(j\omega)$ is an ideal lowpass filter; that is,

$$H_2(j\omega) = \begin{cases} K, & 0 < \omega < W \\ K^*, & -W < \omega < 0, \\ 0, & |\omega| > W \end{cases}$$

where $K$ is a (possibly complex) constant.

**(a)** Find the Fourier transforms of $p(t)$, $y_1(t)$, $y_2(t)$, and $y_3(t)$.

**(b)** Specify the values of $a$, $b$, and $K$ as functions of $\Delta$ such that $z(t) = x(t)$ for any band-limited $x(t)$ and any $\Delta$ such that $0 < \Delta < \pi/W$.

**7.38.** It is frequently necessary to display on an oscilloscope screen waveforms having very short time structures—for example, on the scale of thousandths of a nanosecond. Since the rise time of the fastest oscilloscope is longer than this, such displays cannot be achieved directly. If however, the waveform is periodic, the desired result can be obtained indirectly by using an instrument called a sampling oscilloscope.

The idea, as shown in Figure P7.38(a), is to sample the fast waveform $x(t)$ once each period, but at successively later points in successive periods. The increment $\Delta$ should be an appropriately chosen sampling interval in relation to the bandwidth of $x(t)$. If the resulting impulse train is then passed through an appropriate interpolat-

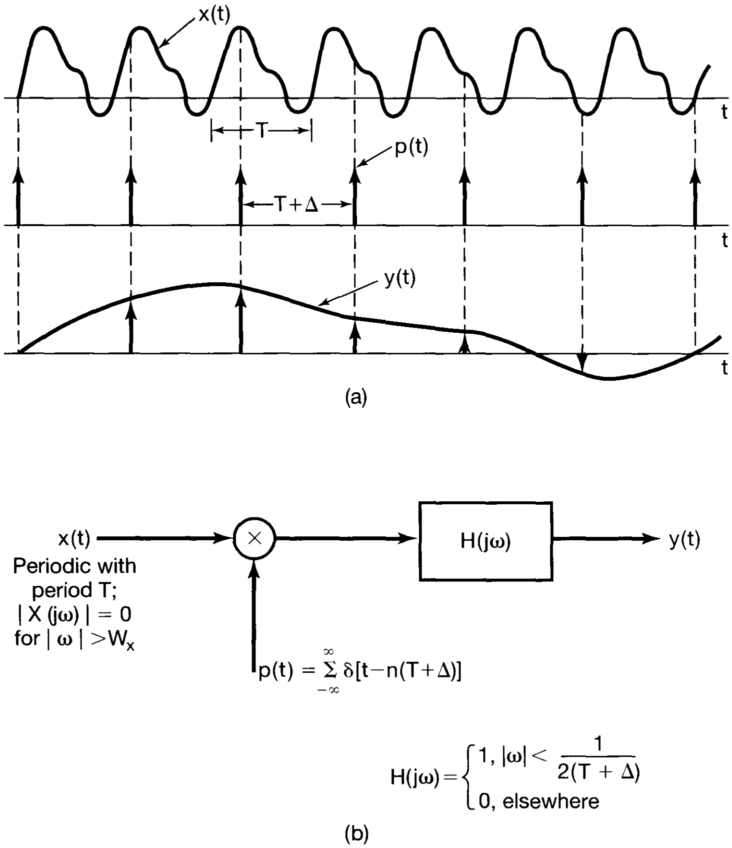



**Figure P7.38**

ing lowpass filter, the output $y(t)$ will be proportional to the original fast waveform slowed down or stretched out in time [i.e., $y(t)$ is proportional to $x(at)$, where $a < 1$].

For $x(t) = A + B\cos[(2\pi/T)t + \theta]$, find a range of values of $\Delta$ such that $y(t)$ in Figure P7.38(b) is proportional to $x(at)$ with $a < 1$. Also, determine the value of $a$ in terms of $T$ and $\Delta$.

**7.39** A signal $x_p(t)$ is obtained through impule-train sampling of a sinusoidal signal $x(t)$ whose frequency is equal to half the sampling frequency $\omega_s$.

$$x(t) = \cos\left(\frac{\omega_s}{2}t + \phi\right)$$

and

$$x_p(t) = \sum_{n=-\infty}^{+\infty} x(nT)\delta(t - nT)$$

where $T = 2\pi/\omega_s$.

**(a)** Find $g(t)$ such that

$$x(t) = \cos(\phi)\cos\left(\frac{\omega_s}{2}t\right) + g(t).$$

**(b)** Show that

$$g(nT) = 0 \qquad \text{for } n = 0, \pm 1, \pm 2, \cdots$$

**(c)** Using the results of the previous two parts, show that if $x_p(t)$ is applied as the input to an ideal lowpass filter with cutoff frequency $\omega_s/2$, the resulting output is

$$y(t) = \cos(\phi)\cos\left(\frac{\omega_s}{2}t\right).$$

**7.40.** Consider a disc on which four cycles of a sinusoid are painted. The disc is rotated at approximately 15 revolutions per second, so that the sinusoid, when viewed through a narrow slit, has a frequency of 60 Hz.

The arrangement is indicated in Figure P7.40. Let $v(t)$ denote the position of the line seen through the slit. Then

$$v(t) = A\cos(\omega_0 t + \phi),\ \omega_0 = 120\pi.$$

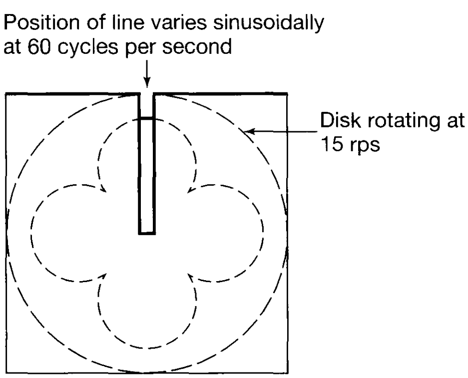



**Figure P7.40**

For notational convenience, we will normalize $v(t)$ so that $A = 1$. At 60 Hz, the eye is not able to follow $v(t)$, and we will assume that this effect can be explained by modeling the eye as an ideal lowpass filter with cutoff frequency 20 Hz.

Sampling of the sinusoid can be accomplished by illuminating the disc with a strobe light. Thus, the illumination can be represented by an impulse train; that is,

$$i(t) = \sum_{k=-\infty}^{+\infty} \delta(t - kT),$$

where $1/T$ is the strobe frequency in hertz. The resulting sampled signal is the product $r(t) = v(t)i(t)$. Let $R(j\omega)$, $V(j\omega)$, and $I(j\omega)$ denote the Fourier transforms of $r(t)$, $v(t)$, and $i(t)$, respectively.

**(a)** Sketch $V(j\omega)$, indicating clearly the effect of the parameters $\phi$ and $\omega_0$.

**(b)** Sketch $I(j\omega)$, indicating the effect of $T$.

**(c)** According to the sampling theorem, there is a maximum value for $T$ in terms of $\omega_0$ such that $v(t)$ can be recovered from $r(t)$ using a lowpass filter. Determine this value of $T$ and the cutoff frequency of the lowpass filter. Sketch $R(j\omega)$ when $T$ is slightly less than the maximum value.

If the sampling period $T$ is made greater than the value determined in part (c), aliasing of the spectrum occurs. As a result of this aliasing, we perceive a lower frequency sinusoid.

**(d)** Suppose that $2\pi/T = \omega_0 + 20\pi$. Sketch $R(j\omega)$ for $|\omega| < 40\pi$. Denote by $v_a(t)$ the apparent position of the line as we perceive it. Assuming that the eye behaves as an ideal lowpass filter with 20-Hz cutoff and unity gain, express $v_a(t)$ in the form

$$v_a(t) = A_a \cos(\omega_a + \phi_a),$$

where $A_a$ is the apparent amplitude, $\omega_a$ the apparent frequency, and $\phi_a$ the apparent phase of $v_a(t)$.

**(e)** Repeat part (d) for $2\pi/T = \omega_0 - 20\pi$.

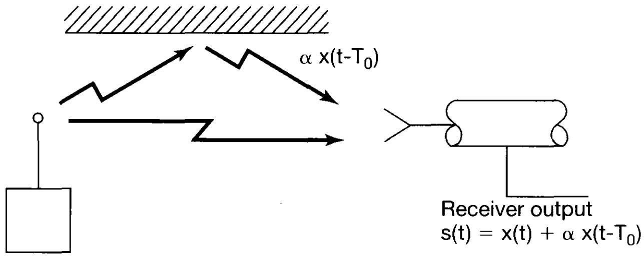
**7.41.** In many practical situations a signal is recorded in the presence of an echo, which we would like to remove by appropriate processing. For example, in Figure P7.41(a), we illustrate a system in which a receiver simultaneously receives a signal $x(t)$ and an echo represented by an attenuated delayed replication of $x(t)$. Thus, the receiver output is $s(t) = x(t) + \alpha x(t - T_0)$, where $|\alpha| < 1$. This output is to be processed to recover $x(t)$ by first converting to a sequence and then using an appropriate digital filter $h[n]$, as indicated in Figure P7.41(b).

Assume that $x(t)$ is band limited [i.e., $X(j\omega) = 0$ for $|\omega| > \omega_M$] and that $|\alpha| < 1$.

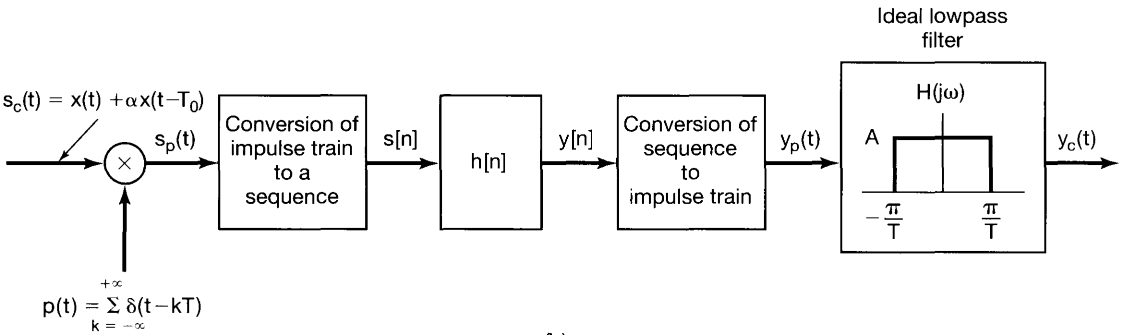
**(a)** If $T_0 < \pi/\omega_M$, and the sampling period is taken to be equal to $T_0$ (i.e., $T = T_0$), determine the difference equation for the digital filter $h[n]$ such that $y_c(t)$ is proportional to $x(t)$.

**(b)** With the assumptions of part (a), specify the gain $A$ of the ideal lowpass filter such that $y_c(t) = x(t)$.

**(c)** Now suppose that $\pi/\omega_M < T_0 < 2\pi/\omega_M$. Determine a choice for the sampling period $T$, the lowpass filter gain $A$, and the frequency response for the digital filter $h[n]$ such that $y_c(t)$ is proportional to $x(t)$.




(a)




(b)

**Figure P7.41**

**7.42.** Consider a band-limited signal $x_c(t)$ that is sampled at a rate higher than the Nyquist rate. The samples, spaced $T$ seconds apart, are then converted to a sequence $x[n]$, as indicated in Figure P7.42.

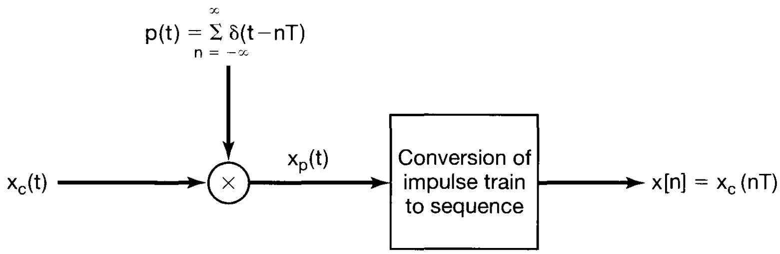



**Figure P7.42**

Determine the relation between the energy $E_d$ of the sequence, the energy $E_c$ of the original signal, and the sampling interval $T$. The energy of a sequence $x[n]$ is defined as

$$E_d = \sum_{n=-\infty}^{\infty} |x[n]|^2,$$

and the energy in a continuous-time function $x_c(t)$ is defined as

$$E_c = \int_{-\infty}^{+\infty} |x_c(t)|^2 \, dt.$$

**7.43.** Figure P7.43(a) depicts a system for which the input and output are discrete-time signals. The discrete-time input $x[n]$ is converted to a continuous-time impulse train $x_p(t)$. The continuous-time signal $x_p(t)$ is then filtered by an LTI system to produce the output $y_c(t)$, which is then converted to the discrete-time signal $y[n]$. The LTI system with input $x_c(t)$ and output $y_c(t)$ is causal and is characterized by the linear constant-coefficient differential equation

$$\frac{d^2y_c(t)}{dt^2} + 4\frac{dy_c(t)}{dt} + 3y_c(t) = x_c(t).$$

The overall system is equivalent to a causal discrete-time LTI system, as indicated in Figure P7.43(b).

Determine the frequency response $H(e^{j\omega})$ and the unit sample response $h[n]$ of the equivalent LTI system.

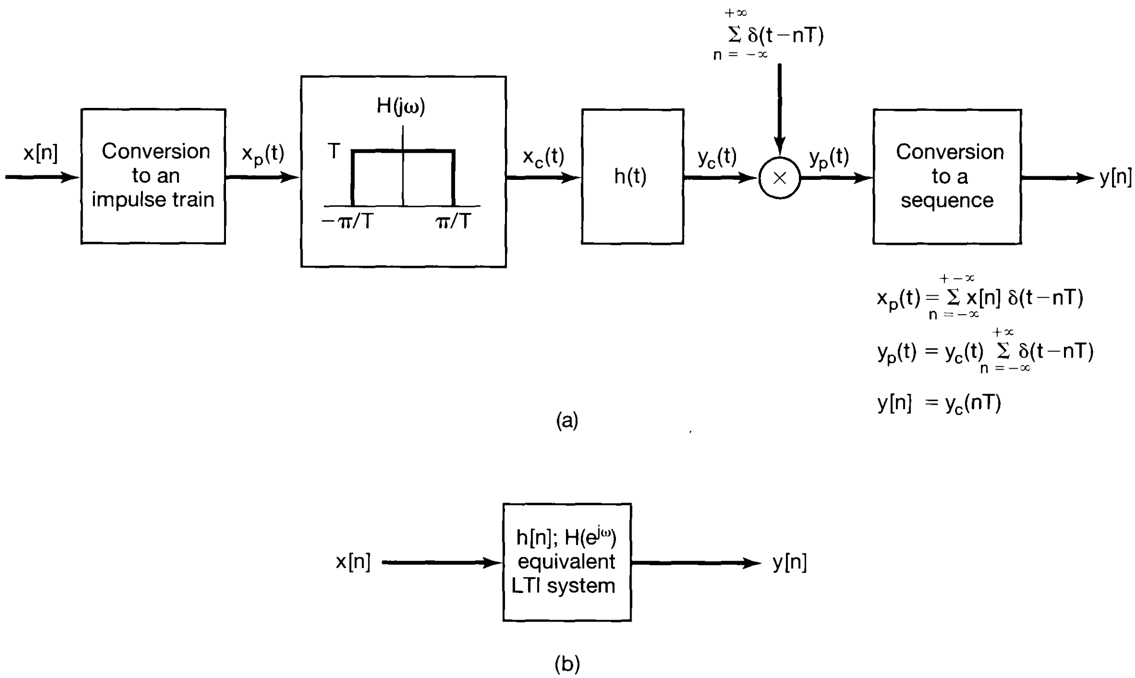



**Figure P7.43**

**7.44.** Suppose we wish to design a continuous-time generator that is capable of producing sinusoidal signals at any frequency satisfying

$$\omega_1 \leq \omega \leq \omega_2,$$

where $\omega_1$ and $\omega_2$ are given positive numbers.

Our design is to take the following form: We have stored a discrete-time cosine wave of period $N$; that is, we have stored $x[0], \ldots, x[N-1]$, where

$$x[k] = \cos\left(\frac{2\pi k}{N}\right).$$

Every $T$ seconds we output an impulse weighted by a value of $x[k]$, where we proceed through the values of $k = 0, 1, \ldots, N-1$ in a cyclic fashion. That is,

$$y_p(kT) = x(k \text{ modulo } N),$$

or equivalently,

$$y_p(kT) = \cos\left(\frac{2\pi k}{N}\right),$$

and

$$y_p(t) = \sum_{k=-\infty}^{+\infty} \cos\left(\frac{2\pi k}{N}\right)\delta(t - kT).$$

**(a)** Show that by adjusting $T$, we can adjust the frequency of the cosine signal being sampled. That is, show that

$$y_p(t) = (\cos\omega_0 t)\sum_{k=-\infty}^{+\infty} \delta(t - kT),$$

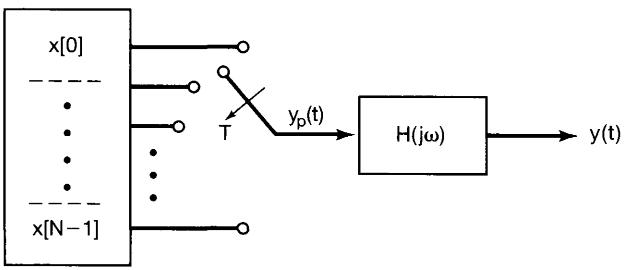
where $\omega_0 = 2\pi/NT$. Determine a range of values for $T$ such that $y_p(t)$ can represent samples of a cosine signal with a frequency that is variable over the full range

$$\omega_1 \leq \omega \leq \omega_2.$$

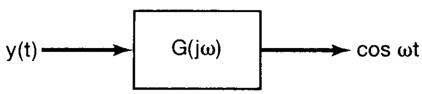
**(b)** Sketch $Y_p(j\omega)$.

The overall system for generating a continuous-time sinusoid is depicted in Figure P7.44(a). $H(j\omega)$ is an ideal lowpass filter with unity gain in its passband; that is,

$$H(j\omega) = \begin{cases} 1, & |\omega| < \omega_c \\ 0, & \text{otherwise} \end{cases}.$$




(a)




(b)

**Figure P7.44**

The parameter $\omega_c$ is to be determined so that $y(t)$ is a continuous-time cosine signal in the desired frequency band.

**(c)** Consider any value of $T$ in the range determined in part (a). Determine the minimum value of $N$ and some value for $\omega_c$ such that $y(t)$ is a cosine signal in the range $\omega_1 \leq \omega \leq \omega_2$.

**(d)** The amplitude of $y(t)$ will vary, depending upon the value of $\omega$ chosen between $\omega_1$ and $\omega_2$. Thus, we must design a system $G(j\omega)$ that normalizes the signal as shown in Figure P7.44(b). Find such a $G(j\omega)$.

**7.45.** In the system shown in Figure P7.45, the input $x_c(t)$ is band limited with $X_c(j\omega) = 0, |\omega| > 2\pi \times 10^4$. The digital filter $h[n]$ is described by the input-output relation

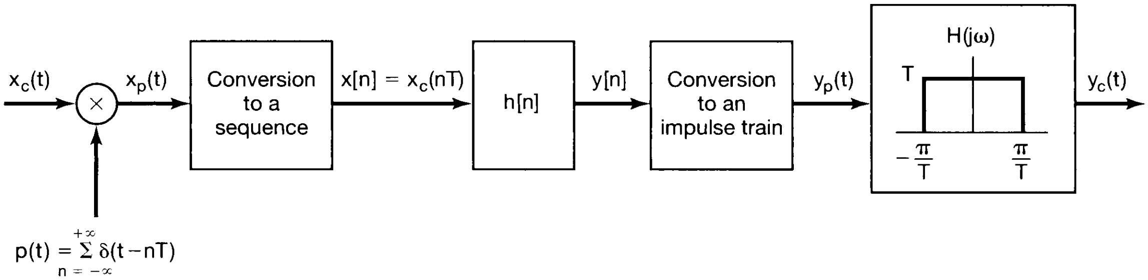



**Figure P7.45**

$$y[n] = T \sum_{k=-\infty}^{n} x[k]. \qquad \text{(P7.45–1)}$$

**(a)** What is the maximum value of $T$ allowed if aliasing is to be avoided in the transformation from $x_c(t)$ to $x_p(t)$.

**(b)** With the discrete-time LTI system specified through eq. (P7.45–1), determine its impulse response $h[n]$.

**(c)** Determine whether there is any value of $T$ for which

$$\lim_{n \to \infty} y[n] = \lim_{t \to \infty} \int_{-\infty}^{t} x_c(\tau)\, d\tau. \qquad \text{(P7.45–2)}$$

If so, determine the *maximum* value. If not, explain and specify how $T$ would be chosen so that the equality in eq. (P7.45–2) is best approximated. (Think carefully about this part; it is easy to jump to the wrong conclusion!)

**7.46** A signal $x[n]$ is sampled in discrete time as shown in Figure P7.46. $h_r[n]$ is an ideal lowpass filter with frequency response

$$H_r(e^{j\omega}) = \begin{cases} 1, & |\omega| < \frac{\pi}{N} \\ 0, & \frac{\pi}{N} < |\omega| < \pi \end{cases}$$

From eqs. (7.46) and (7.47), the filter output is expressible as

$$x_r[n] = \sum_{k=-\infty}^{+\infty} x[kN]h_r[n-kN] = \sum_{k=-\infty}^{+\infty} x[kN]\frac{N\omega_c}{\pi}\frac{\sin \omega_c(n-kN)}{\omega_c(n-kN)}$$

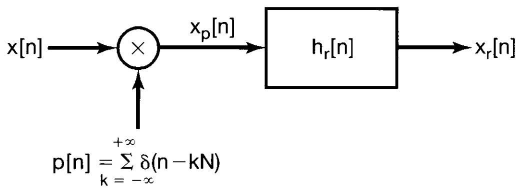



**Figure P7.46**

where $\omega_c = 2\pi/N$. Show that independent of whether the sequence $x[n]$ is sampled above or below the Nyquist rate, $x_r[mN] = x[mN]$, where $m$ is any positive or negative integer.

**7.47.** Suppose $x[n]$ has a Fourier transform that is zero for $\pi/3 \le |\omega| \le \pi$. Show that

$$x[n] = \sum_{k=-\infty}^{\infty} x[3k]\left(\frac{\sin(\frac{\pi}{3}(n-3k))}{\frac{\pi}{3}(n-3k)}\right).$$

**7.48.** If $x[n] = \cos(\frac{\pi}{4}n+\phi_0)$ with $0 \le \phi_0 < 2\pi$ and $g[n] = x[n]\sum_{k=-\infty}^{\infty}\delta[n-4k]$, what additional constraints must be imposed on $\phi_0$ to ensure that

$$g[n] * \left(\frac{\sin\frac{\pi}{4}n}{\frac{\pi}{4}n}\right) = x[n]?$$

**7.49.** As discussed in Section 7.5 and illustrated in Figure 7.37, the procedure for interpolation or upsampling by an integer factor $N$ can be thought of as the cascade of two operations. The first operation, involving system $A$, corresponds to inserting $N-1$ zero-sequence values between each sequence value of $x[n]$, so that

$$x_p[n] = \begin{cases} x_d\left[\frac{n}{N}\right], & n = 0, \pm N, \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}.$$

For exact band-limited interpolation, $H(e^{j\omega})$ is an ideal lowpass filter.

**(a)** Determine whether or not system $A$ is linear.

**(b)** Determine whether or not system $A$ is time invariant.

**(c)** For $X_d(e^{j\omega})$ as sketched in Figure P7.49 and with $N = 3$, sketch $X_p(e^{j\omega})$.

**(d)** For $N = 3$, $X_d(e^{j\omega})$ as in Figure P7.49, and $H(e^{j\omega})$ appropriately chosen for exact band-limited interpolation, sketch $X(e^{j\omega})$.

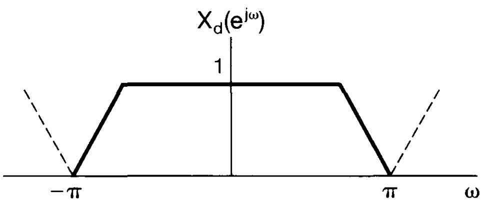



**Figure P7.49**

**7.50.** In this problem, we consider the discrete-time counterparts of the zero-order hold and first-order hold, which were discussed for continuous time in Sections 7.1.2 and 7.2.

Let $x[n]$ be a sequence to which discrete-time sampling, as illustrated in Figure 7.31, has been applied. Suppose the conditions of the discrete-time sampling theorem are satisfied; that is, $\omega_s > 2\omega_M$, where $\omega_s$ is the sampling frequency and $X(e^{j\omega}) = 0$, $\omega_M < |\omega| \leq \pi$. The original signal $x[n]$ is then exactly recoverable from $x_p[n]$ by ideal lowpass filtering, which, as discussed in Section 7.5, corresponds to band-limited interpolation.

The zero-order hold represents an approximate interpolation whereby every sample value is repeated (or held) $N - 1$ successive times, as illustrated in Figure P7.50(a) for the case of $N = 3$. The first-order hold represents a linear interpolation between samples, as illustrated in the same figure.

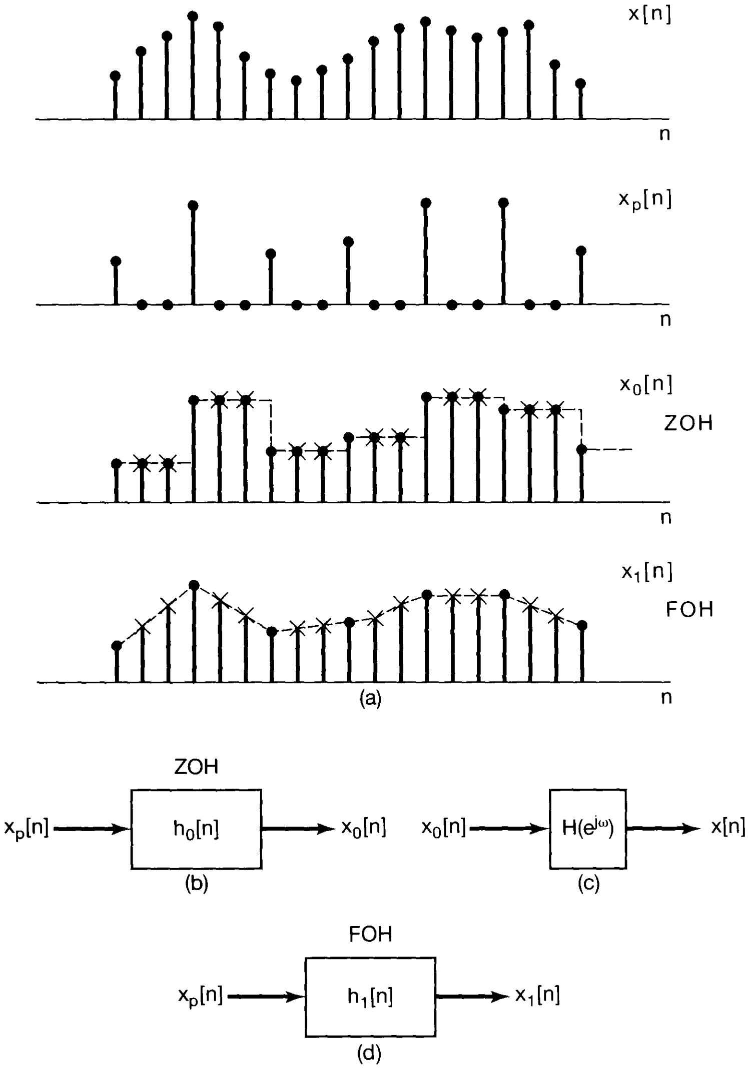



Figure P7.50

**(a)** The zero-order hold can be represented as an interpolation in the form of eq. (7.47) or, equivalently, the system in Figure P7.50(b). Determine and sketch $h_0[n]$ for the general case of a sampling period $N$.

**(b)** $x[n]$ can be exactly recovered from the zero-order-hold sequence $x_0[n]$ using an appropriate LTI filter $H(e^{j\omega})$, as indicated in Figure P7.50(c). Determine and sketch $H(e^{j\omega})$.

**(c)** The first-order-hold (linear interpolation) can be represented as an interpolation in the form of eq. (7.47) or, equivalently, the system in Figure P7.50(d). Determine and sketch $h_1[n]$ for the general case of a sampling period $N$.

**(d)** $x[n]$ can be exactly recovered from the first-order-hold sequence $x_1[n]$ using an appropriate LTI filter with frequency response $H(e^{j\omega})$. Determine and sketch $H(e^{j\omega})$.

**7.51.** As shown in Figure 7.37 and discussed in Section 7.5.2, the procedure for interpolation or upsampling by an integer factor $N$ can be thought of as a cascade of two operations. For exact band-limited interpolation, the filter $H(e^{j\omega})$ in Figure 7.37 is an ideal lowpass filter. In any specific application, it would be necessary to implement an approximate lowpass filter. In this problem, we explore some useful constraints that are often imposed on the design of these approximate lowpass filters.

**(a)** Suppose that $H(e^{j\omega})$ is approximated by a zero-phase FIR filter. The filter is to be designed with the constraint that the original sequence values $x_d[n]$ get reproduced *exactly;* that is,

$$x[n] = x_d\left[\frac{n}{L}\right], n = 0, \pm L, \pm 2L, \ldots. \qquad \text{(P7.51–1)}$$

This guarantees that, even though the interpolation between the original sequence values may not be exact, the original values are reproduced exactly in the interpolation. Determine the constraint on the impulse response $h[n]$ of the lowpass filter which guarantees that eq. (P7.51–1) will hold exactly for any sequence $x_d[n]$.

**(b)** Now suppose that the interpolation is to be carried out with a *linear-phase, causal, symmetric* FIR filter of length $N$; that is

$$h[n] = 0, n < 0, n > N - 1, \qquad \text{(P7.51–2)}$$

$$H(e^{j\omega}) = H_R(e^{j\omega})e^{-j\alpha\omega}, \qquad \text{(P7.51–3)}$$

where $H_R(e^{j\omega})$ is real. The filter is to be designed with the constraint that the original sequence values $x_d[n]$ get reproduced *exactly,* but with an integer delay $\alpha$, where $\alpha$ is the negative of the slope of the phase of $H(e^{j\omega})$; that is,

$$x[n] = x_d\left[\frac{n - \alpha}{L}\right], n - \alpha = 0, \pm L, \pm 2L, \ldots \qquad \text{(P7.51–4)}$$

Determine whether this imposes any constraint on whether the filter length $N$ is odd or even.

**(c)** Again, suppose that the interpolation is to be carried out with a linear-phase, causal, symmetric FIR filter, so that

$$H(e^{j\omega}) = H_R(e^{j\omega})e^{-j\beta\omega},$$

where $H_R(e^{j\omega})$ is real. The filter is to be designed with the constraint that the original sequence values $x_d[n]$ get reproduced exactly, but with a delay $M$ that is not necessarily equal to the slope of the phase; that is,

$$x[n] = x_d\left[\frac{n-\alpha}{L}\right], n - M = 0, \pm L, \pm 2L, \ldots.$$

Determine whether this imposes any constraint on whether the filter length $N$ is odd or even.

**7.52** In this problem we develop the dual to the time-domain sampling theorem, whereby a time-limited signal can be reconstructed from *frequency-domain* samples. To develop this result, consider the frequency-domain sampling operation in Figure P7.52.

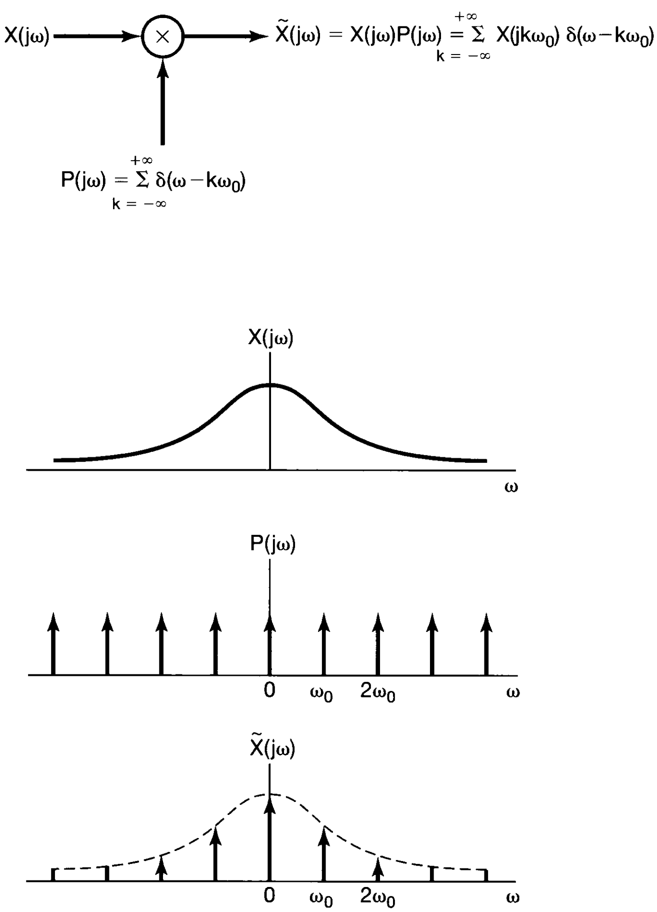



Figure P7.52

**(a)** Show that

$$\tilde{x}(t) = x(t) * p(t)$$

where $\tilde{x}(t)$, $x(t)$, and $p(t)$ are the inverse Fourier transforms of $\tilde{X}(j\omega)$, $X(j\omega)$, and $P(j\omega)$, respectively.

**(b)** Assuming that $x(t)$ is time-limited so that $x(t) = 0$ for $|t| \geq \frac{\pi}{\omega_o}$, show that $x(t)$ can be obtained from $\tilde{x}(t)$ through a "low-time windowing" operation. That is,

$$x(t) = \tilde{x}(t)w(t)$$

where

$$w(t) = \begin{cases} \omega_0, & |t| \leq \frac{\pi}{\omega_0} \\ 0, & |t| > -\frac{\pi}{\omega_0} \end{cases}$$

**(c)** Show that $x(t)$ is not recoverable from $\tilde{x}(t)$ if $x(t)$ is not constrained to be zero for $|t| \geq \frac{\pi}{\omega_0}$.