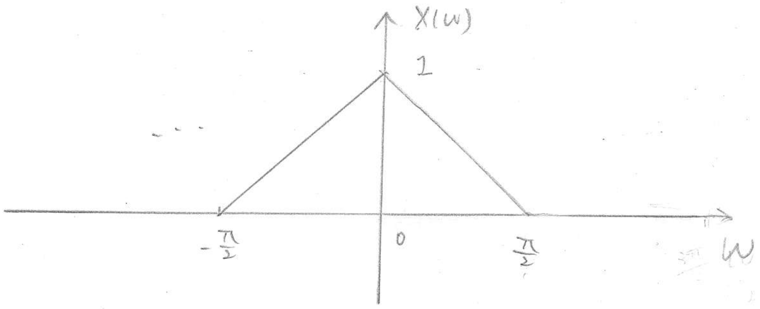
ECE 438 HW 4 Solution

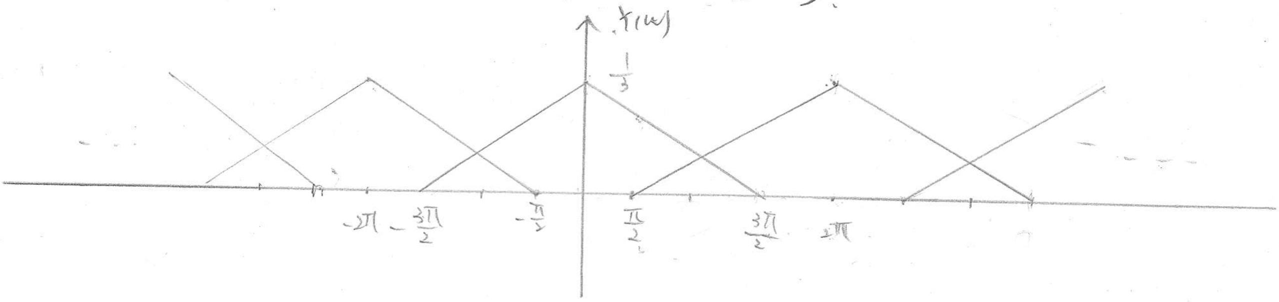
1. a.

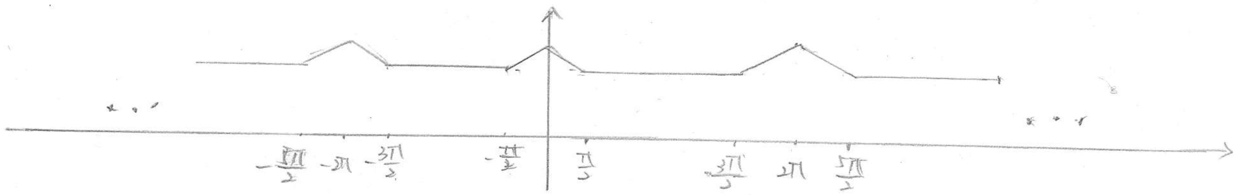
$$X(w) = \begin{cases} 1 - |w|/(\frac{\pi}{2}) & |w| \le \frac{\pi}{2} \\ 0 & \frac{\pi}{2} \le |w| \le \pi . \end{cases}$$



b. $y[n] = x[Dn] \longleftrightarrow Y(w) = \frac{1}{D} \sum_{k=0}^{D-1} X\left(\frac{w - 2\pi k}{D}\right)$

if $D=3$, then $Y(w) = \frac{1}{3} \sum_{k=0}^{2} X\left(\frac{w - 2\pi k}{3}\right)$.



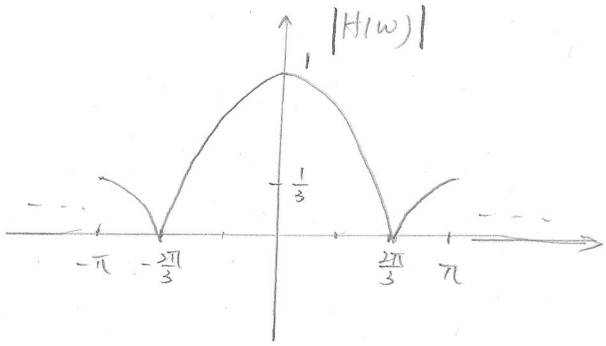


2. a. $y[n] = \frac{1}{3}(x[n] + x[n-1] + x[n-2])$

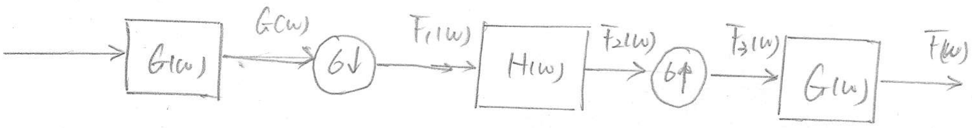
$$Y(w) = \frac{1}{3}(X(w) + e^{-jw}X(w) + e^{-2jw}X(w))$$

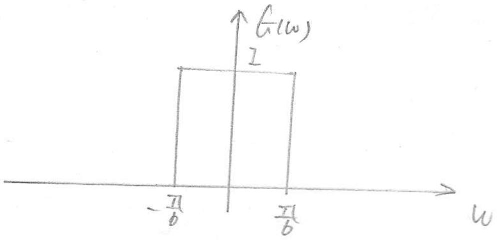
$$H(w) = \frac{Y(w)}{X(w)} = \frac{1}{3}(1 + e^{-jw} + e^{-2jw})$$

b.

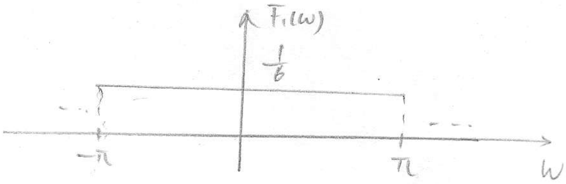


c/d.

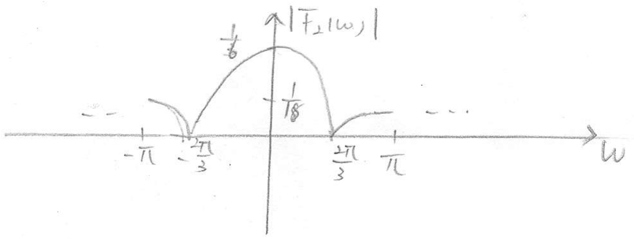




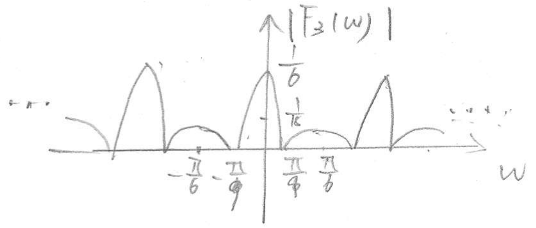
$$F_1(w) = \frac{1}{6}\sum_{k=0}^{5} G\left(\frac{w-2\pi k}{6}\right)$$



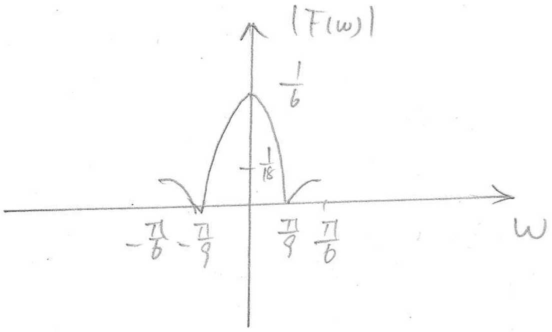
$$F_2(w) = F_1(w)\,H(w) = \frac{1}{18}(1+e^{-jw} + e^{-2jw})$$



$F_3(w) = F_2(6w)$



$$F(w) = F_3(w)\,G(w) = \begin{cases} \frac{1}{18}\left(1 + e^{-j6w} + e^{-12jw}\right) & |w| \le \frac{\pi}{6} \\ 0 & \pi > |w| > \frac{\pi}{6} \end{cases}$$



e. $h[n]$ is easy to design with three δ in digital filters.

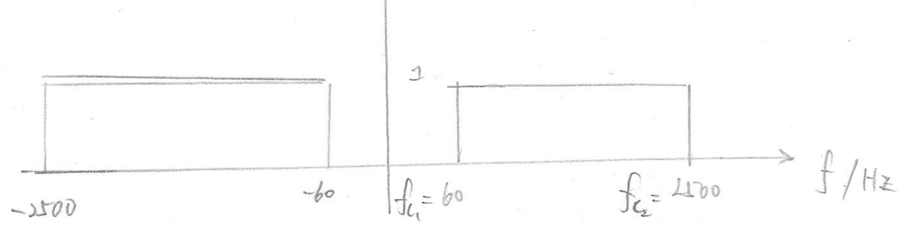
$F(w) = \frac{1}{6} H(w) \cdot \mathrm{rect}\left(\frac{w}{\frac{\pi}{3}}\right)$

$f[n] = \frac{1}{6} h[n] * \mathrm{sinc}$ is hard to design.

3. a. Need a bandpass filter or high pass filter with $f_{c_1} = 60\,Hz$

$f_{c_2} = 2500\,Hz$

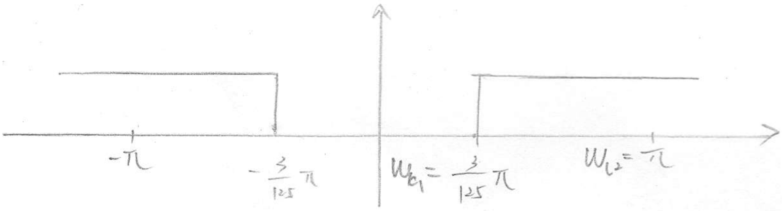
ideal filter



Select sampling frequence at 5KHz

$$W_{c_1} = 2\pi \frac{f_{c_1}}{f_s} = \frac{3}{125}\pi$$

$$W_{c_2} = 2\pi \frac{f_{c_2}}{f_s} = \pi$$

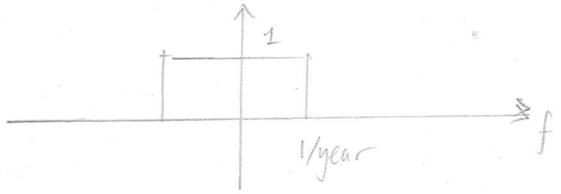


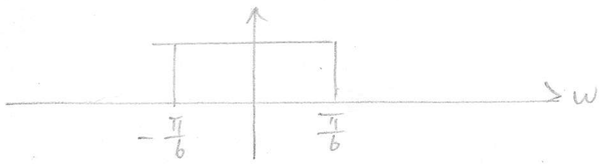
b. Need a low pass filter applied to monthly data

$f_c =$ 1 cycle / year

$f_s =$ 12 samples / year.

$$w_c = 2\pi \frac{f_c}{f_s} = \frac{\pi}{6}$$





4. $x(t) = 0.8\cos[2\pi(500)t] + 0.3\cos(2\pi(3500)t]$

$x[n] = x(\frac{n}{f_s}) = 0.8\cos[2\pi\frac{500n}{10k}] + 0.3\cos[2\pi\frac{3500}{10k}n]$

$= 0.8\cos\frac{\pi}{10}n + 0.3\cos\frac{7\pi}{10}n$

$X_{512}(k) = X_{tr}(w)\Big|_{w=\frac{2\pi k}{512}}$ for $k \in [0, 511]$

$X_{tr}(w) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(w-u)W(u)\,du$

where $X(w) = DTFT\{x[n]\} = 0.8\pi\, rep_{2\pi}[\delta(w+\frac{\pi}{10}) + \delta(w-\frac{\pi}{10})]$
$+ 0.3\pi\, rep_{2\pi}[\delta(w+\frac{7}{10}\pi) + \delta(w-\frac{7\pi}{10})]$

and $W(w) = DTFT\{u[n] - u[n-512]\}$

$= \frac{\sin(w\frac{512}{2})}{\sin(\frac{w}{2})} e^{-jw\frac{512-1}{2}}$

$= \frac{\sin(256w)}{\sin\frac{w}{2}} e^{-jw\frac{511}{2}}$

$W(w)$ has maximum at $w=0$ when $W(0) = 512$

$\therefore X_{tr}(w) = \frac{0.8\pi}{2\pi} rep_{2\pi}[W(w+\frac{\pi}{10}) + W(w-\frac{\pi}{10})]$

$+ \frac{0.3\pi}{2\pi} rep_{2\pi}[W(w+\frac{7}{10}\pi) + W(w-\frac{7}{10}\pi)]$

$X_{tr}(w)$ has peaks at $w = \pm\frac{\pi}{10} + 2\pi n$
$w = \pm\frac{7\pi}{10} + 2\pi n$, $n \in \mathbb{Z}$

Since $X_{512}(k) = X_{tr}(w)\Big|_{w=\frac{2\pi k}{512}}$

$X_{512}(k)$ has peaks at $\frac{2\pi k}{512} = \pm\frac{\pi}{10} + 2\pi n$. and $\pm\frac{7}{10}\pi + 2\pi n$,

$\frac{2\pi k_1}{512} = \frac{\pi}{10} \Rightarrow k_1 = 25.6$

$$\frac{2\pi k_2}{512} = 2\pi - \frac{\pi}{10} \Rightarrow k_2 = 486.4$$

$$\frac{2\pi k_3}{512} = \frac{7}{10}\pi \Rightarrow \quad k_3 = 179.2$$

$$\frac{2\pi k_4}{512} = 2\pi - \frac{7}{10}\pi \Rightarrow k_4 = 332.8$$

Since k is integer $\Rightarrow$ X(k) has peaks at 26, 486, 179, 333.

$$|X(k)|_{k=26,\,486} \approx \frac{0.8\pi}{2\pi} 512 = 204.8$$

$$|X(k)|_{k=179,\,333} \approx \frac{0.3\pi}{2\pi} \cdot 512 = 76.8$$

5. $y[n] = \begin{cases} x[n] & n = 0, \cdots, N-1 \\ 0 & n = N, \cdots, M-1 \end{cases}$

$$Y[k] = DFT_M\{y[n]\} = \sum_{n=0}^{M-1} y[n]\, e^{-j\frac{2\pi k n}{M}}$$

$$= \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi k n}{M}} + \sum_{n=N}^{M-1} 0 \cdot e^{-j\frac{2\pi k n}{M}}$$

$$= \sum_{n=0}^{N-1} x[n]\, e^{-j\frac{2\pi k n}{M}}$$

$$= \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\frac{2\pi k n}{M}} \qquad (\text{Because } x[n] = 0 \text{ for } n \notin [0, N-1])$$

Compared with $X(w) = DTFT(x[n]) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-jwn}$

$$\Rightarrow Y[k] = X(w)\Big|_{w = \frac{2\pi k}{M}} \qquad \text{for } k = 0, 1, \cdots M-1.$$