ECE 301

Division 1, Fall 2006

Instructor: Mimi Boutin

Final Examination

Instructions:

1. Wait for the "BEGIN" signal before opening this booklet. In the meantime, read the instructions below and fill out the requested info.

2. You have two hours to answer the 8 questions contained in this exam, for a total of up to 147 points. **When the end of the exam is announced, you must stop writing immediately**. Anyone caught writing after the exam is over will get a grade of zero.

3. This booklet contains 19 pages. The last pages contain a table of formulas and properties (5 pages) as well as some scratch paper (4 pages). You may detach the scratch paper and the formula pages from the booklet **once the exam begins**. Each transform and each property is labeled with a number. To save time, you may use these numbers to specify which transform/property you are using when justifying your answer. In general, if you use a fact which is *not* contained in this table, you must explain why it is true in order to get full credit. The only exception are the properties of the ROC, which you can use without justification.

4. This is a closed book exam. The only personal items allowed are pens/pencils, erasers and something to drink. Anything else is strictly forbidden. That includes cell phones, iPods, and PDAs.

5. You must keep your eyes on your exam at all times. Looking around is strictly forbidden.

Name:____________________________________

Email:____________________________________

Signature:_________________________________

(15 pts) **1.** An LTI system has unit impulse response $h[n] = u[n+2]$. Compute the system's response to the input $x[n] = \left(\frac{1+j}{3}\right)^n u[n]$. (Simplify your answer until all $\sum$ signs disappear.)

2. A discrete-time system is such that when the input is one of the signals in the left column, then the output is the corresponding signal in the right column:

$$
\begin{array}{rcl}
\text{input} & & \text{output} \\
x_0[n] = \delta[n] & \rightarrow & y_0[n] = \delta[n-1], \\
x_1[n] = \delta[n-1] & \rightarrow & y_1[n] = 4\delta[n-2], \\
x_2[n] = \delta[n-2] & \rightarrow & y_2[n] = 9\delta[n-3], \\
x_3[n] = \delta[n-3] & \rightarrow & y_3[n] = 16\delta[n-4], \\
& \vdots & \\
x_k[n] = \delta[n-k] & \rightarrow & y_k[n] = (k+1)^2\delta[n-(k+1)] \text{ for any integer k.}
\end{array}
$$

(10 pts) **a)** Can this system be time-invariant? Explain.

(10 pts) **b)** Assuming that this system is linear, what input $x[n]$ would yield the output $y[n] = u[n-1]$?

(22 pts) **3.** Let $x(t)$ and $y(t)$ be the input and the output of a discrete-time system, respectively. Answer each of the questions below with either yes or no (no justification needed).

	Yes	No
If $y(t) = x(2t)$, is the system causal?	☐	☐
If $y(t) = (t+2)x(t)$, is the system causal?	☐	☐
If $y(t) = x(-t^2)$, is the system causal?	☐	☐
If $y(t) = x(t) + t - 1$, is the system memoryless?	☐	☐
If $y(t) = x(t^2)$, is the system memoryless?	☐	☐
If $y(t) = x(t/3)$, is the system stable?	☐	☐
If $y(t) = tx(t/3)$, is the system stable?	☐	☐
If $y(t) = \int_{-\infty}^{t} x(\tau)d\tau$, is the system stable?	☐	☐
If $y(t) = \sin(x(t))$, is the system time invariant?	☐	☐
If $y(t) = u(t) * x(t)$, is the system LTI?	☐	☐
If $y(t) = (tu(t)) * x(t)$, is the system linear?	☐	☐

(15 pts) **4** Compute the energy and the power of the signal $x(t) = \frac{3e^{jt}}{1+j}$.

(15 pts) **5.** Compute the coefficients a_k of the Fourier series of the signal

$$x(t) = \sum_{k=-\infty}^{\infty} 3\left(u(t+\frac{1}{2}+2k) - u(t-\frac{1}{2}+2k)\right).$$

(Simplify your answer as much as possible.)

(15 pts) **6.** The Laplace transform of the unit impulse response of a system is

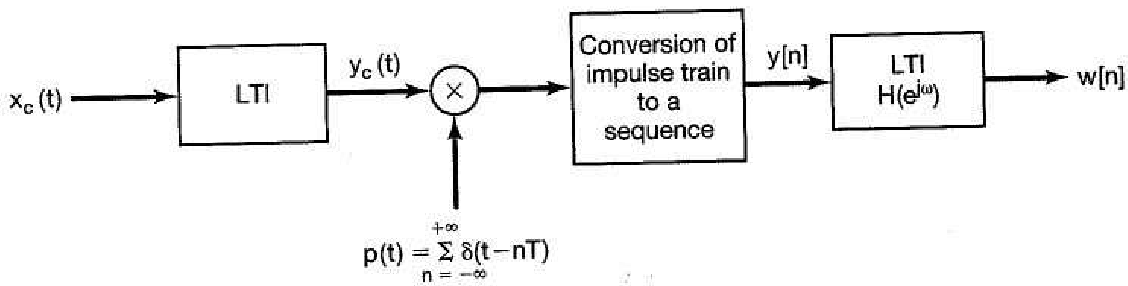
$$H(s) = \frac{1}{s+2}, \mathrm{Re}(s) > 2.$$

Determine the response $y(t)$ of the system when the input is $x(t) = e^{-3|t|}$.

7. The block diagram below shows a system consisting of a continuous-time LTI system followed by a sampler, conversion to a sequence, and an LTI discrete-time system. The continuous-time LTI system is causal and satisfies the linear, constant-coefficient differential equation

$$\frac{dy_c(t)}{dt} + y_c(t) = x_c(t).$$

The input $x_c(t)$ is a unit impulse $\delta(t)$.



(10 pts) **a)** Determine the input $y_c(t)$.

(Problem 7 continues on the next page.)

(15 pts) **b)** Determine the frequency response $H(e^{j\omega})$ and the unit impulse response $h[n]$ such that $w[n] = \delta[n]$.

8. A commonly used system to maintain privacy in voice communication is a *speech scrambler*. The input of the system is a normal speech signal $x(t)$ and the output is the scrambled version $y(t)$. The signal $y(t)$ is transmitted and then unscrambled at the receiver.

We assume that all inputs to the scrambler are real and band limited to the frequency ω_M; that is, $\mathcal{X}(\omega) = 0$ for $|\omega| > \omega_M$. Given any such input, our proposed scrambler permutes different bands of the spectrum of the input signal. In addition, the output is real and band limited to the same frequency band: that is $\mathcal{Y}(\omega) = 0$ for $|\omega| > \omega_M$. The specific algorithm for the scrambler is

$$\begin{aligned} \mathcal{Y}(\omega) &= \mathcal{X}(\omega - \omega_M), \text{ when } \omega > 0, \\ \mathcal{Y}(\omega) &= \mathcal{X}(\omega + \omega_M), \text{ when } \omega < 0. \end{aligned}$$

(5 pts) **a)** Assuming that $\mathcal{X}(\omega) = \omega^2(u(\omega+3) - u(\omega-3))$, sketch the graph of $\mathcal{Y}(\omega)$.

(10 pts) **b)** Draw a block diagram for such an ideal scrambler.

(10 pts) **c)** Draw a block diagram for the associated unscrambler.

Table

DT Signal Energy and Power

$$E_\infty = \sum_{n=-\infty}^{\infty} |x[n]|^2 \tag{1}$$

$$P_\infty = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2 \tag{2}$$

CT Signal Energy and Power

$$E_\infty = \int_{-\infty}^{\infty} |x(t)|^2 \, dt \tag{3}$$

$$P_\infty = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 \, dt \tag{4}$$

Fourier Series of CT Periodic Signals with period T

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\left(\frac{2\pi}{T}\right)t} \tag{5}$$

$$a_k = \frac{1}{T} \int_0^T x(t) e^{-jk\left(\frac{2\pi}{T}\right)t} dt \tag{6}$$

Fourier Series of DT Periodic Signals with period N

$$x[n] = \sum_{k=0}^{N-1} a_k e^{jk\left(\frac{2\pi}{N}\right)n} \tag{7}$$

$$a_k = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-jk\left(\frac{2\pi}{N}\right)n} \tag{8}$$

CT Fourier Transform

$$\text{F.T.}: X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \tag{9}$$

$$\text{Inverse F.T.}: x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega \tag{10}$$

Properties of CT Fourier Transform

Let $x(t)$ be a continuous-time signal and denote by $X(\omega)$ its Fourier transform. Let $y(t)$ be another continuous-time signal and denote by $Y(\omega)$ its Fourier transform.

	Signal	*FT*	
Linearity:	$ax(t) + by(t)$	$aX(\omega) + bY(\omega)$	(11)
Time Shifting:	$x(t - t_0)$	$e^{-j\omega t_0}X(\omega)$	(12)
Frequency Shifting:	$e^{j\omega_0 t}x(t)$	$X(\omega - \omega_0)$	(13)
Time and Frequency Scaling:	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{\omega}{a}\right)$	(14)
Multiplication:	$x(t)y(t)$	$\frac{1}{2\pi}X(\omega) * Y(\omega)$	(15)
Convolution:	$x(t) * y(t)$	$X(\omega)Y(\omega)$	(16)
Differentiation in Time:	$\frac{d}{dt}x(t)$	$j\omega X(\omega)$	(17)

Some CT Fourier Transform Pairs

$$e^{j\omega_0 t} \xrightarrow{\mathcal{F}} 2\pi\delta(\omega - \omega_0) \tag{18}$$

$$1 \xrightarrow{\mathcal{F}} 2\pi\delta(\omega) \tag{19}$$

$$\frac{\sin Wt}{\pi t} \xrightarrow{\mathcal{F}} u(\omega + W) - u(\omega - W) \tag{20}$$

$$u(t + T_1) - u(t - T_1) \xrightarrow{\mathcal{F}} \frac{2\sin(\omega T_1)}{\omega} \tag{21}$$

$$\delta(t) \xrightarrow{\mathcal{F}} 1 \tag{22}$$

$$u(t) \xrightarrow{\mathcal{F}} \frac{1}{j\omega} + \pi\delta(\omega) \tag{23}$$

$$e^{-at}u(t), \mathcal{R}e\{a\} > 0 \xrightarrow{\mathcal{F}} \frac{1}{a + j\omega} \tag{24}$$

$$te^{-at}u(t), \mathcal{R}e\{a\} > 0 \xrightarrow{\mathcal{F}} \frac{1}{(a + j\omega)^2} \tag{25}$$

DT Fourier Transform

Let $x[n]$ be a discrete-time signal and denote by $X(\omega)$ its Fourier transform.

$$\text{F.T.:} X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \tag{26}$$

$$\text{Inverse F.T.: } x[n] = \frac{1}{2\pi}\int_{2\pi} X(\omega)e^{j\omega n}\,d\omega \tag{27}$$

Properties of DT Fourier Transform

Let $x(t)$ be a signal and denote by $X(\omega)$ its Fourier transform. Let $y(t)$ be another signal and denote by $Y(\omega)$ its Fourier transform.

	Signal	*F.T.*	
Linearity:	$ax[n] + by[n]$	$aX(\omega) + bY(\omega)$	(28)
Time Shifting:	$x[n - n_0]$	$e^{-j\omega n_0} X(\omega)$	(29)
Frequency Shifting:	$e^{j\omega_0 n} x[n]$	$X(\omega - \omega_0)$	(30)
Time Reversal:	$x[-n]$	$X(-\omega)$	(31)
Time Exp.:	$x_k[n] = \begin{cases} x[\frac{n}{k}], & \text{if } k \text{ divides } n \\ 0, & \text{else.} \end{cases}$	$X(\omega)$	(32)
Multiplication:	$x[n]y[n]$	$\frac{1}{2\pi} X(\omega) * Y(\omega)$	(33)
Convolution:	$x[n] * y[n]$	$X(\omega)Y(\omega)$	(34)
Differencing in Time:	$x[n] - x[n-1]$	$(1 - e^{-j\omega})X(\omega)$	(35)

Some DT Fourier Transform Pairs

$$\sum_{k=0}^{N-1} a_k e^{jk\left(\frac{2\pi}{N}\right)n} \xrightarrow{\mathcal{F}} 2\pi \sum_{k=-\infty}^{\infty} a_k \delta(\omega - \frac{2\pi k}{N}) \tag{36}$$

$$e^{j\omega_0 n} \xrightarrow{\mathcal{F}} 2\pi \sum_{l=-\infty}^{\infty} \delta(\omega - \omega_0 - 2\pi l) \tag{37}$$

$$1 \xrightarrow{\mathcal{F}} 2\pi \sum_{l=-\infty}^{\infty} \delta(\omega - 2\pi l) \tag{38}$$

$$\frac{\sin Wn}{\pi n}, 0 < W < \pi \xrightarrow{\mathcal{F}} u(\omega + W) - u(\omega - W) X(\omega) = \begin{cases} 1, & 0 \le |\omega| < W \\ 0, & \pi \ge |\omega| > W \end{cases} \tag{39}$$

$X(\omega)$periodic with period 2π

$$\delta[n] \xrightarrow{\mathcal{F}} 1 \tag{40}$$

$$u[n] \xrightarrow{\mathcal{F}} \frac{1}{1 - e^{-j\omega}} + \pi \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k) \tag{41}$$

$$\alpha^n u[n], |\alpha| < 1 \xrightarrow{\mathcal{F}} \frac{1}{1 - \alpha e^{-j\omega}} \tag{42}$$

$$(n+1)\alpha^n u[n], |\alpha| < 1 \xrightarrow{\mathcal{F}} \frac{1}{(1 - \alpha e^{-j\omega})^2} \tag{43}$$

Laplace Transform

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt \tag{44}$$

Properties of Laplace Transform

Let $x(t)$, $x_1(t)$ and $x_2(t)$ be three CT signals and denote by $X(s)$, $X_1(s)$ and $X_2(s)$ their respective Laplace transform. Let R be the ROC of $X(s)$, let R_1 be the ROC of $X_1(z)$ and let R_2 be the ROC of $X_2(s)$.

	Signal	L.T.	ROC	
Linearity:	$ax_1(t) + bx_2(t)$	$aX_1(s) + bX_2(s)$	At least $R_1 \cap R_2$	(45)
Time Shifting:	$x(t - t_0)$	$e^{-st_0}X(s)$	R	(46)
Shifting in s:	$e^{s_0 t}x(t)$	$X(s - s_0)$	$R + s_0$	(47)
Conjugation:	$x^*(t)$	$X^*(s^*)$	R	(48)
Time Scaling:	$x(at)$	$\frac{1}{\lvert a\rvert} X\left(\frac{s}{a}\right)$	aR	(49)
Convolution:	$x_1(t) * x_2(t)$	$X_1(s)X_2(s)$	At least $R_1 \cap R_2$	(50)
Differentiation in Time:	$\frac{d}{dt}x(t)$	$sX(s)$	At least R	(51)
Differentiation in s:	$-tx(t)$	$\frac{dX(s)}{ds}$	R	(52)
Integration :	$\int_{-\infty}^{t} x(\tau)d\tau$	$\frac{1}{s}X(s)$	At least $R \cap \mathcal{R}e\{s\} > 0$	(53)

Some Laplace Transform Pairs

Signal	*LT*	*ROC*	
$\delta(t)$	1	all s	(54)
$u(t)$	$\frac{1}{s}$	$\mathcal{R}e\{s\} > 0$	(55)
$u(t)\cos(\omega_0 t)$	$\frac{s}{s^2 + \omega_0^2}$	$\mathcal{R}e\{s\} > 0$	(56)
$e^{-\alpha t}u(t)$	$\frac{1}{s + \alpha}$	$\mathcal{R}e\{s\} > -\alpha$	(57)
$-e^{-\alpha t}u(-t)$	$\frac{1}{s + \alpha}$	$\mathcal{R}e\{s\} < -\alpha$	(58)
			(59)

z-Transform

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} \tag{60}$$

Properties of z-Transform

Let $x[n]$, $x_1[n]$ and $x_2[n]$ be three DT signals and denote by $X(z)$, $X_1(z)$ and $X_2(z)$ their respective z-transform. Let R be the ROC of $X(z)$, let R_1 be the ROC of $X_1(z)$ and let R_2 be the ROC of $X_2(z)$.

	Signal	z-T.	ROC	
Linearity:	$ax_1[n] + bx_2[n]$	$aX_1(z) + bX_2(z)$	At least $R_1 \cap R_2$	(61)
Time Shifting:	$x[n - n_0]$	$z^{-n_0}X(z)$	R, but perhaps adding/deleting $z = 0$	(62)
Time Shifting:	$x[-n]$	$X(z^{-1})$	R^{-1}	(63)
Scaling in z:	$e^{j\omega_0 n}x[n]$	$X(e^{-j\omega_0}z)$	R	(64)
Conjugation:	$x^*(t)$	$X^*(z^*)$	R	(65)
Convolution:	$x_1[n] * x_2[n]$	$X_1(z)X_2(z)$	At least $R_1 \cap R_2$	(66)

Some z-Transform Pairs

Signal	LT	ROC	
$u[n]$	$\frac{1}{1 - z^{-1}}$	$\|z\| > 1$	(67)
$-u(-n - 1)$	$\frac{1}{1 - z^{-1}}$	$\|z\| < 1$	(68)
$\alpha^n u[n]$	$\frac{1}{1 - \alpha z^{-1}}$	$\|z\| > \alpha$	(69)
$-\alpha^n u[-n - 1]$	$\frac{1}{1 - \alpha z^{-1}}$	$\|z\| < \alpha$	(70)
$\delta[n]$	1	all z	(71)

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