(15 pts) **1.** Using the definition of the Fourier transform, compute the Fourier transform of the DT signal:

$$x[n] = e^{j\frac{\pi}{17}n}\left(\frac{1}{3}\right)^n u[n-1].$$

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} e^{j\frac{\pi}{17}n}\left(\frac{1}{3}\right)^n u[n-1] e^{-j\omega n}$$, n must be ≥ 1

$$= \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n e^{-jn(\omega - \frac{\pi}{17})}$$, let $m = n-1$

$$= \frac{1}{3} \cdot e^{-j(\omega - \frac{\pi}{17})} \sum_{m=0}^{\infty} \left(\frac{1}{3}\right)^m \underbrace{e^{-jm(\omega - \frac{\pi}{17})}}_{|u| = 1}$$

$$= \boxed{\frac{e^{-j(\omega - \frac{\pi}{17})}}{3 - e^{-j(\omega - \pi/17)}}}$$

(15 pts) **2.** Using the definition of the inverse Fourier transform, compute the CT inverse Fourier transform of

$$\mathcal{X}(\omega) = \sum_{k=-\infty}^{\infty} \frac{1}{2^k} u[k]\delta(\omega - k\pi).$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \frac{1}{2^k} u[k] \delta(\omega - k\pi) e^{j\omega t} d\omega$$

$$= \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} \left(\frac{1}{2}\right)^k u[k] \delta(\omega - k\pi) e^{j\omega t} d\omega \quad ,$$

- sift integral when $\omega = k\pi$
- k must be ≥ 0

$$= \frac{1}{2\pi}\sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k \underbrace{e^{jk\pi t}}_{|u| = 1}$$

$$= \boxed{\frac{1}{2\pi} \cdot \frac{1}{1 - \frac{1}{2}e^{j\pi t}}}$$

(15 pts) **3.** Given is a DT signal $x[n] = j\cos(g[n])$ where $g[n]$ is a real signal and an odd function of n.

a) Bob claims that the Fourier transform of $x[n]$ is $\mathcal{X}(\omega) = \frac{j}{\sin\omega}$. Explain why Bob's answer is wrong.

$x[n]$ is imag & even

$$\mathcal{X}(-\omega) = \frac{j}{\sin(-\omega)} = \frac{j}{-\sin(\omega)} \text{ odd}$$

$\mathcal{X}(\omega)$ is imag & odd, so $x[n]$ must be real & odd

$x[n]$ is imag. & even

$\therefore$ Bob is wrong

b) Alice says that the Fourier transform of $x[n]$ is $\mathcal{X}(\omega) = \frac{3}{\cos\omega}$. Could Alice be right? Explain.

$$\mathcal{X}(-\omega) = \frac{3}{\cos(-\omega)} = \frac{3}{\cos(\omega)} \text{ even}$$

$\mathcal{X}(\omega)$ is real & even $\rightarrow$ $x[n]$ must be real & even

$x[n]$ is imag. & even

$\therefore$ Alice is wrong.

c) Devin says that the Fourier transform of $x[n]$ is $\mathcal{X}(\omega) = \frac{j}{(\omega^2+1)^2}$. Could Devin be right? Explain.

$$\mathcal{X}(-\omega) = \frac{j}{((-\omega)^2+1)^2} = \frac{j}{(\omega^2+1)^2} \text{ even}$$

$\mathcal{X}(\omega)$ is imag. & even

$$\mathcal{X}(\omega) \stackrel{?}{=} \mathcal{X}^*(-\omega)$$

$$\mathcal{X}^*(-\omega) = \frac{-j}{((-\omega)^2+1)^2} = \frac{-j}{(\omega^2+1)^2} \neq \mathcal{X}(\omega)$$

$\therefore$ Devin could be right

4. A discrete-time LTI system is defined by the difference equation

$$y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = 2x[n]$$

(10 pts) a) Find the frequency response of this system. (Use the properties of the Fourier transform listed in the table to justify your answer. Do not just plug into a formula you have learned by heart.)

$$\mathcal{F}[y[n]] - \frac{3}{4}e^{-j\omega} \cdot \mathcal{F}[y[n]] + \frac{1}{8}e^{-2j\omega} \cdot \mathcal{F}[y[n]] = 2\mathcal{F}[x[n]]$$

$$Y(\omega) = \frac{2X(\omega)}{\left(1 - \frac{3}{4}e^{-j\omega} + \frac{1}{8}e^{-2j\omega}\right)} = X(\omega) \cdot H(\omega)$$

$$\therefore \quad H(\omega) = \frac{2}{1 - \frac{3}{4}e^{-j\omega} + \frac{1}{8}e^{-2j\omega}} = \frac{16}{(e^{-j\omega})^2 - 6(e^{-j\omega}) + 8}$$

$$= \frac{16}{(e^{-j\omega} - 2)(e^{-j\omega} - 4)}$$

$$= \frac{A/2}{1 - \frac{1}{2}e^{-j\omega}} + \frac{B/4}{1 - \frac{1}{4}e^{-j\omega}}$$

$Ae^{-j\omega} - 2A + Be^{-j\omega} - 4B = 16$

$\omega = 0, \quad -A - 3B = 16$

$$\therefore \boxed{h[n] = A/2\left(\tfrac{1}{2}\right)^n u[n] + B/4\left(\tfrac{1}{4}\right)^n u[n]}$$

(10 pts) b) What is the unit impulse response of this system. (Justify your answer)

$$y[n] = h[n] * \delta[n] = h[n]$$

$$= \boxed{\left[\frac{A}{2}\left(\frac{1}{2}\right)^n + \frac{B}{4}\left(\frac{1}{4}\right)^n\right]u[n]}$$

(5 pts) c) What is the Fourier transform of the output when the input is $x[n] = \left(\frac{1}{4}\right)^n u[n]$? (Justify your answer.)

$$x[n] = \left(\frac{1}{4}\right)^n u[n] \xrightarrow{\mathcal{F}} X(\omega) = \frac{1}{1-\frac{1}{4}e^{-j\omega}} = \frac{-4}{-4+e^{-j\omega}}$$

$$Y(\omega) = H(\omega) \cdot X(\omega) = \frac{64}{(e^{-j\omega}-2)(e^{-j\omega}-4)^2}$$

$$= \boxed{\frac{-A/2}{(1-\frac{1}{2}e^{-j\omega})} + \frac{-B/4}{(1-\frac{1}{4}e^{-j\omega})} + \frac{-C/4}{(1-\frac{1}{4}e^{-j\omega})^2}}$$

(10 pts) d) What is the output when the input is $x[n] = \left(\frac{1}{4}\right)^n u[n]$? (Justify your answer)

$$\boxed{y[n] = \left[-\frac{A}{2}\left(\frac{1}{2}\right)^n - \frac{B}{4}\left(\frac{1}{4}\right)^n - \frac{C}{4}(n+1)\left(\frac{1}{4}\right)^n\right]u[n]}$$

(15 pts) **5.** Use the properties of the Fourier transform to evaluate the following integral:

$$\int_{-\infty}^{\infty} \frac{\sin^2(4t)}{t^2} dt.$$

$$\frac{\sin Wt}{\pi t} \xrightarrow{\mathcal{F}} u(\omega+W) - u(\omega-W)$$

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$$

$$\Rightarrow \pi^2 \int_{-\infty}^{\infty} \left|\frac{\sin 4t}{\pi t}\right|^2 dt = \frac{\pi}{2}\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$$

$$= \frac{\pi}{2}\int_{-\infty}^{\infty} [u(\omega+4) - u(\omega-4)] d\omega$$

$$= \frac{\pi}{2}\int_{-4}^{4} d\omega$$

$$= \boxed{4\pi}$$