26 JANUARY 2012

PROBABILITY

Definition: A sample spase S is a collection of "samples/outpust" s.

ex

Rolling a die has a sample space $S = \{1, 2, 3, 4, 5, 6\}$

A discrete probability space (in contrast to continuous) together with probability function/distribution p subject to following cond.

(1) $p(s)$ is always real and between 0 and 1

(2) $\sum_{s \in S} p(s) = 1$

NOTE: In a continuous probability state in which $|S| = \infty$, instead of $\sum_{s \in S} p(s)$, condition $\int p(s)\, ds = 1$ ...

If $p(s) = p(s')$ for all s and s', then it is in equidistribution

eg.

→ rolling of fair die implies that $p(s) = \frac{1}{6}$, hence equidistribution

More generally, if S has finitely many elements, equidistribution implies $p(s) = \frac{1}{|S|}$

On a sample space with $|S| = \infty$, a discrete equidistribution cannot exist. (though ~~think~~ $p(s) = 0$ for continuous sample space S ?)

Ex The order

Ex "Pick at random a natural number and give each natural number the same chance"

cannot be exeuted.

Def: An event E is a collection of outcomes inside a given sample space. E is a subset of S.

for any event E

$$P(E) = \sum_{s \in E} p(s)$$

Ex Roll 2 fair dice

$S' = \{\text{sum of rolled dice}\}$

$= \{2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$

Then, graphically, the ~~funct~~ probability dist:

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

each square has chance of $\frac{1}{36}$

$\Rightarrow$ NOT an equidistribution since

$p(2) = 1/36$, $p(3) = 2/36$, ... $p(8) = 5/36$ ...

$p(12) = 1/36$.

Q: What are probabilities for

E_1 = an even sum

E_2 = sum of at least 8

E_3 = even sum ≥ 8 ?

A: $E_1 = \{2, 4, 8, 10, 12\}$

$$p(E_1) = \sum_{e \in E_1} p(e) = p(2) + p(4) + p(8) + p(10) + p(12) + p(6)$$

$$= \frac{1 + 3 + 5 + 5 + 3 + 1}{36} = \boxed{\frac{1}{2}}$$

$\rightarrow$ This is not an obvious result: there are more even outcomes and each outcome has a ~~d~~ different probability

$E_2 = \{8, 9, 10, 11, 12\}$

$$p(E_2) = p(8) + p(9) + p(10) + p(11) + p(12) = \frac{5+4+3+2+1}{36} = \frac{5}{12}$$

$E_3 = \{8, 10, 12\}$

$$p(E_3) = \frac{5+3+1}{36} = \frac{1}{4}$$

$\Rightarrow$ We would like to establish some association between $p(E_1)$, $p(E_2)$ and $p(E_3)$, namely if I know $p(E_1)$ and $p(E_2)$, can I deduce $p(E_3)$ where $E_3 = E_1 \cap E_2$?

"YES" indicates that E_1 and E_2 unrelated.

"No" would suggest correlation

→ To illustrate more dramatically, we may be interested in observing association / pattern in

Q_2: sum ≥ 9

Q_1: sum ≥ 8

Would answer to Q_2 reveal something about Q_1?

Returning to our example;

$$P(E_1) \cdot P(E_2) = P(E_3)$$
$$\frac{1}{2} \cdot \frac{5}{12} \neq \frac{1}{4}$$

But consider a sample space comprising only of E_2. Then the 5 outcomes $\{8, 9, 10, 11, 12\}$ would now have readjusted their probability so that

$$\sum p(e) = 1 \text{ still holds.}$$

but readjusted so that $p(8) : p(9) : p(10) : p(11) : p(12)$ (ratio still same)

In this restricted sample space, we can find that

$p(12) = \frac{1}{15}$ $p(11) = \frac{2}{15}$ ... $p(8) = \frac{5}{15}$

Now we ask the question: What is p(even roll) = $p(E_3$, in new sample space$)$

$$p(\text{even role}) = p(12) + p(10) + p(8) = \frac{1+3+5}{15} = \frac{9}{15} = \frac{3}{5}$$

comparing with p(even role) as addressed in E_1, we see that

$p(\text{even}, E_3) = \frac{3}{5} > \frac{1}{2}$. Thus it is E_1, E_2 are not an indp event.

Def 2 events E and F are independent if $p(E \cap F) = p(E) \cdot p(F)$.

→ Knowing whether or not an outcome B is in E does not indicate anything at all of the outcome in F

Def $P(E|F) = \dfrac{P(E \cap F)}{P(F)}$, conditional prob E given F.

= This would be the prob for E to occur, provided that we shrink the sample space ~~transect~~ changed from S to F.

Returning for our example,

$$P(E_2 | E_1) = \frac{P(E_1 \cap E_2)}{P(E_2)} = \frac{\frac{1}{4}}{\frac{5}{12}} = \frac{3}{5}$$, which is what we calc above.

Q: If you make ~~the~~ a list of birthdays of n people, and we ask for the chance of 2 people having the same birthday, ~~whe~~ for what n does this chance exceed $\frac{1}{2}$?

FACT For any event E, $\bar{E}$ denotes complementary event ($s \in \bar{E}$, then $s \notin E$)

Approach: How do we have n distinct birthdays?

↳ think of a person giving out birthdays...

1st person: $p = 1$ (any birthday would do)

2nd person: $p = \frac{364}{365}$ (choice rest to $365 - 1$)

3rd person: $p = \frac{363}{365}$

$\vdots$

nth person: $p = \frac{365 - (n-1)}{365}$

$$p(n \text{ diff birthdays}) = \prod_{i=1}^{n} \frac{365 - i + 1}{365}$$

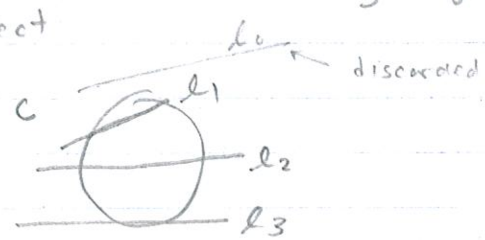
↳ eventually, p will fall below $\frac{1}{2}$ then $\bar{p}$ will exceed $\frac{1}{2}$.

Turns out $n \geq 23$. (cool)

PREVIEW OF NEXT LESSON

Given a circle C in a plane, we randomly drop line ℓ into the plane, discarding any cases for which C & ℓ do not intersect

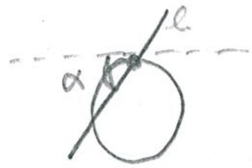
So



If all lines have the same chance, find the prob. that the second is longer than the radius.

Answer, 1

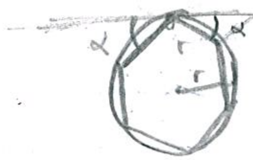
→ Rotate the diagram so that the one point of intersection faces north



→ draw a horizontal tanget line for that point

→ Now, we classify each l with its angle α. all angles have the same chance.

recall



regular hexagon fits inside a circle has side length r.
then, angles $|a| > \alpha$ will have secant greatr than r.

180/6

1080

6 | 1080

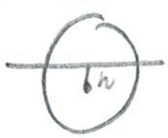
Since $\alpha = 60$; and tanget line maks 180

$\frac{120}{180} = \frac{2}{3}$

$p(E) = \frac{2}{3}$

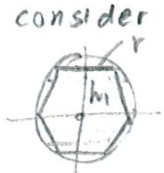
Answer 2

Rotate diagram so l is horizontal



all heights h have same probability.

consider



any height greatr than h_1 will have secant < radius.

→ we find $h_1 = \frac{\sqrt{3}}{2} r$. Consider circle w/ $r = 1$,

prob(secant > radius) = $\frac{\sqrt{3}}{2}$

Diff??

he will explain