2. Properties of Impulse Response and Dirac Delta Func.

1) Some implications of Linearity & Time-Invariance

Given : $x(t) \longrightarrow \overset{LTI}{\boxed{h(t)}} \longrightarrow y(t) = x(t) * h(t)$

TI implies :

$$x(t-t_0) \longrightarrow \boxed{h(t)} \longrightarrow y(t-t_0) = x(t-t_0) * h(t)$$

Also : $x(t-t_0) * h(t) = y(t-t_0)$

$x(t) * h(t-t_0) = y(t-t_0)$

and Also : $a x(t-t_1) * b h(t-t_2) = ab\, y(t-(t_1+t_2))$

"Linearity"

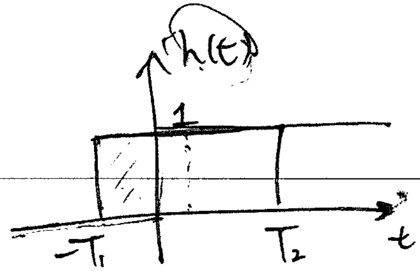
example : Integrator

$$y(t) = \int_{t-T_2}^{t+T_1} x(\tau)\, d\tau \quad , \quad T_1, T_2 \geq 0$$

- Impulse response ? $(h(t))$

$$h(t) = \int_{t-T_2}^{t+T_1} \delta(\tau)\, d\tau = \begin{cases} 1 & \text{if } \begin{pmatrix} t+T_1 > 0 \\ t-T_2 < 0 \end{pmatrix} \Leftrightarrow \begin{pmatrix} t > -T_1 \\ t < T_2 \end{pmatrix} \Leftrightarrow -T_1 < t < T_2 \\ 0 & \text{o.w.} \end{cases}$$

* o.w. = otherwise



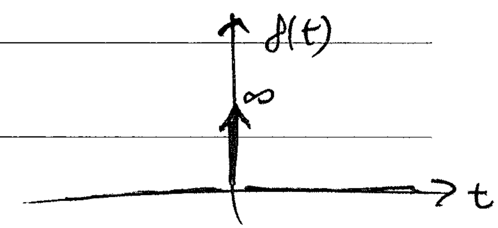
- causal? No, unless $T_1 = 0$
- stable? Yes $\int_{-\infty}^{\infty} |h(t)|\, dt < \infty$

as long as $T_1, T_2 < \infty$

- special case: $T_1 = 0$, and $T_2 = \infty$

$$y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau \Rightarrow h(t) = u(t) \text{ : unit step}$$

$\Rightarrow$ causal but not stable



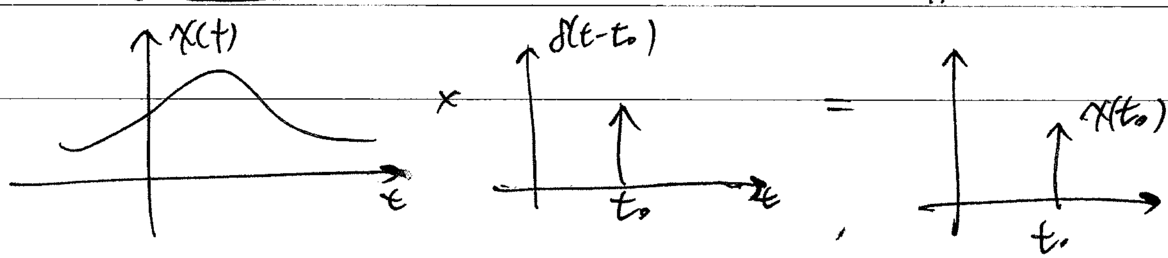
2) Properties of Delta Func.

- Recall properties that we learned :

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1$$

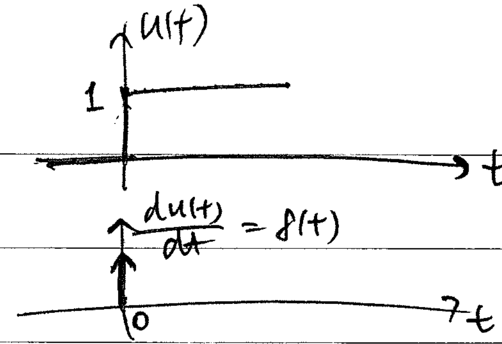
$$\int_{-\infty}^{\infty} \delta(t-t_0) = \int_{t_0^-}^{t_0^+} \delta(t-t_0)\, dt = 1$$

$$\underline{x(t)\,\delta(t-t_0) = x(t_0)\,\delta(t-t_0)}$$



Symmetric : $\delta(-t) = \delta(t)$

derivative at discontinuity : $\delta(t) = \dfrac{du(t)}{dt}$



- Additional Properties of Delta Func.

$$x(t) * \delta(t) = x(t)$$

$$x(t) * \delta(t-t_0) = x(t-t_0)$$

(proof)

$$x(t) * \delta(t-t_0) = \int_{-\infty}^{\infty} \delta(\tau - t_0)\, x(t-\tau)\, d\tau$$

multiplication property of $\delta(t)$

$$= \int_{-\infty}^{\infty} \delta(\tau - t_0)\, x(t-t_0)\, d\tau$$

$x(t-t_0)$ does not depend on τ

$$= x(t-t_0) \underbrace{\int_{-\infty}^{\infty} \delta(\tau - t_0)\, d\tau}_{=1}$$

$$= x(t-t_0)$$

//

- Similarly can easily show :

$$\delta(t-t_1) * \delta(t-t_2) = \delta(t-(t_1+t_2))$$

Prob. 2.11 $\quad x(t) \longrightarrow \overset{\text{LTI}}{\boxed{h(t)}} \longrightarrow y(t)$

- $\dfrac{dx(t)}{dt} \longrightarrow \boxed{h(t)} \longrightarrow z(t)$. Show $z(t) = \dfrac{dy(t)}{dt}$

$x(t) \longrightarrow \boxed{\dfrac{d}{dt}} \longrightarrow \dfrac{dx(t)}{dt}$: is an LTI system

So, we have a system :

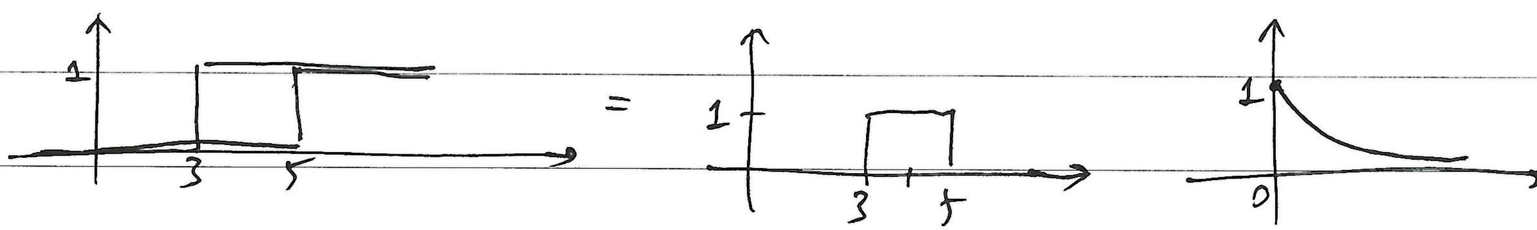
① $x(t) \longrightarrow \boxed{\dfrac{d}{dt}} \longrightarrow \boxed{h(t)} \longrightarrow z(t)$

Invoking associativity & commutativity :

② $x(t) \longrightarrow \boxed{h(t)} \longrightarrow \boxed{\dfrac{d}{dt}} \longrightarrow z(t)$

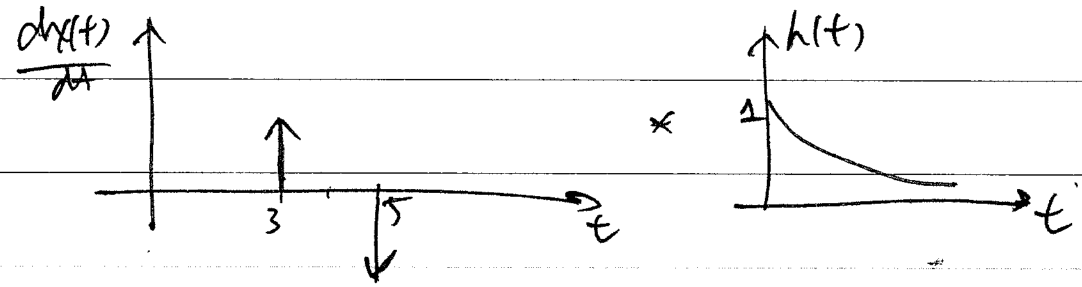
Thus; $z(t) = \dfrac{dy(t)}{dt}$ //

- $x(t) = u(t-3) - u(t-5) = \text{rect}\left(\dfrac{t-4}{2}\right)$, $h(t) = e^{-3t}u(t)$



Suppose you were interested in $z(t) = \frac{dy(t)}{dt}$ where $y(t) = x(t) * h(t)$

It's certainly easier to first differentiate $x(t)$ and then convolve with $h(t)$.



$$* \quad = z(t)$$

$$h(t) = e^{-3t} u(t)$$

$$z(t) = \frac{d}{dt} y(t) = h(t-3) - h(t-5) = e^{-3(t-3)} u(t-3) - e^{-3(t-5)} u(t-5)$$

$$\Rightarrow \quad y(t) = x(t) * h(t) = h(t) * x(t)$$

$$= e^{-3t} u(t) * \{u(t-3) - u(t-5)\}$$

$$= e^{-3t} u(t) * \{u(t) * \delta(t-3) - u(t) * \delta(t-5)\}$$

$$= e^{-3t} u(t) * u(t) * \{\delta(t-3) - \delta(t-5)\}$$

(will be shown later)

$$= \frac{1}{3}(1 - e^{-3t}) u(t) * \{\delta(t-3) - \delta(t-5)\}$$

distributive

$$= \frac{1}{3}\left(1 - e^{-3(t-3)}\right) u(t-3) - \frac{1}{3}\left(1 - e^{-3(t-5)}\right) u(t-5)$$