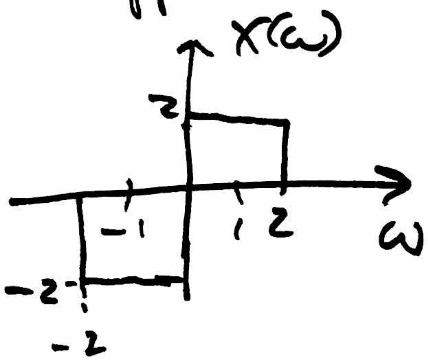
4.4b

Find $x(t)$ given $X(\omega) = \begin{cases} 2, & 0 \le \omega \le 2 \\ -2, & -2 \le \omega \le 0 \\ 0 & \text{else} \end{cases}$

$\rightarrow$ Direct approach: run the integration



$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

$$= \frac{1}{2\pi}\int_{-2}^{0} (-2) e^{j\omega t} d\omega + \frac{1}{2\pi}\int_{0}^{2} (2) e^{j\omega t} d\omega$$

$$= -\frac{1}{\pi} \cdot \frac{1}{jt}\left[e^{j\omega t}\right]_{-2}^{0} + \frac{1}{\pi} \cdot \frac{1}{jt}\left[e^{j\omega t}\right]_{0}^{2}$$

$$= \frac{1}{\pi} \cdot \frac{1}{jt}\left(-\left(1 - e^{-j2t}\right) + e^{j2t} - 1\right)$$

$$= \frac{1}{j\pi t}\left(2\cos 2t - 2\right) = \frac{2}{j\pi t}\left(\cos 2t - 1\right)$$

$$\sin^2\theta = \frac{1}{2} - \frac{1}{2}\cos(2\theta)$$

$$= -\frac{4}{j\pi t}\sin^2 t = \frac{4j}{\pi t}\sin^2 t$$

Using modulation:

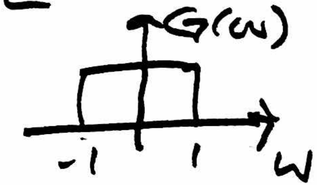
$$X(\omega) = G(\omega-\omega_0) - G(\omega+\omega_0)$$

Then we can use the properties:

$$\frac{\sin Wt}{\pi t} \xleftrightarrow{CTFT} X(\omega) = \begin{cases} 1 & |\omega| < W \\ 0 & \text{else} \end{cases}$$

$$x(t)\cdot \sin\omega_0 t \xleftrightarrow{CTFT} \frac{1}{2j}\left[X(\omega-\omega_0) - X(\omega+\omega_0)\right]$$

Then $G(\omega) = \begin{cases} 2, & |\omega| < 1 \\ 0, & \text{else} \end{cases}$



Then $g(t) = 2\,\frac{\sin t}{\pi t}$

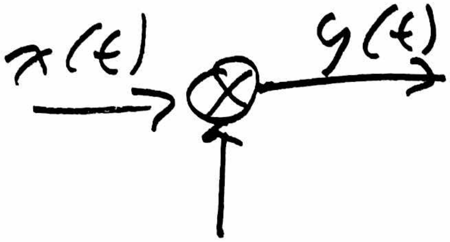
$$\mathcal{F}^{-1}\{G(\omega-\omega_0) - G(\omega+\omega_0)\} = 2j\,g(t)\sin\omega_0 t$$

$\omega_0 = 1$

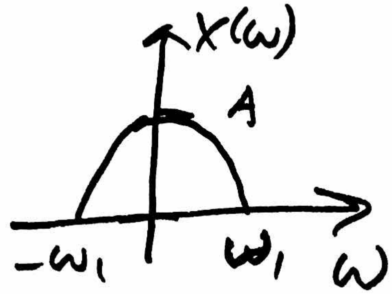
$$= 2j \cdot 2\,\frac{\sin(t)}{\pi t}\cdot \sin 1t$$

$$= \frac{4j}{\pi t}\sin^2 t$$

Ex Demodulation



$$Y(\omega) = \frac{1}{2}\left(X(\omega-\omega_0) + X(\omega+\omega_0)\right)$$



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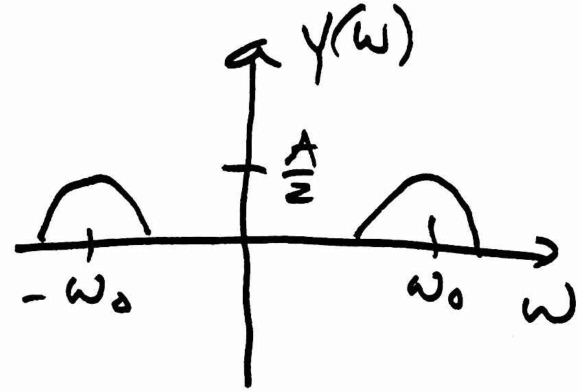
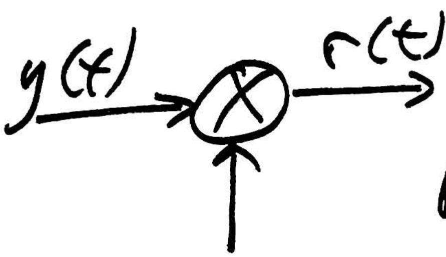


How do we recover $x(t)$?

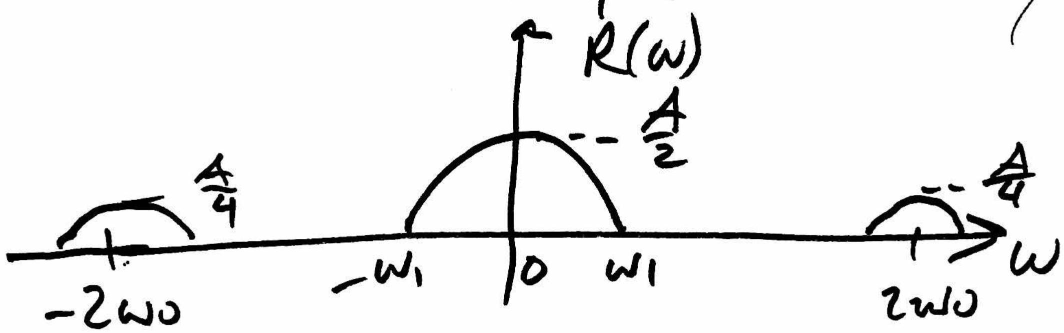


$r(t)$ ~ recieved

$$R(\omega) = \frac{1}{2}\left(Y(\omega-\omega_0) + Y(\omega+\omega_0)\right)$$

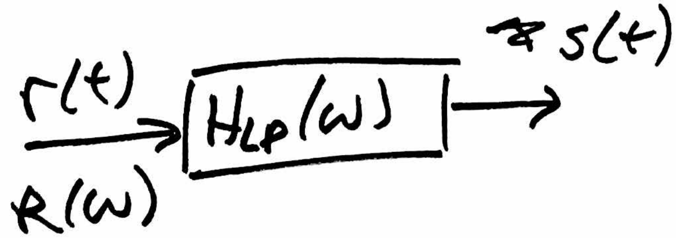
$$= \frac{1}{4}\left(X(\omega-2\omega_0) + X(\omega) + X(\omega) + X(\omega+2\omega_0)\right)$$

$$= \frac{1}{4}\left(X(\omega-2\omega_0) + 2X(\omega) + X(\omega+2\omega_0)\right)$$

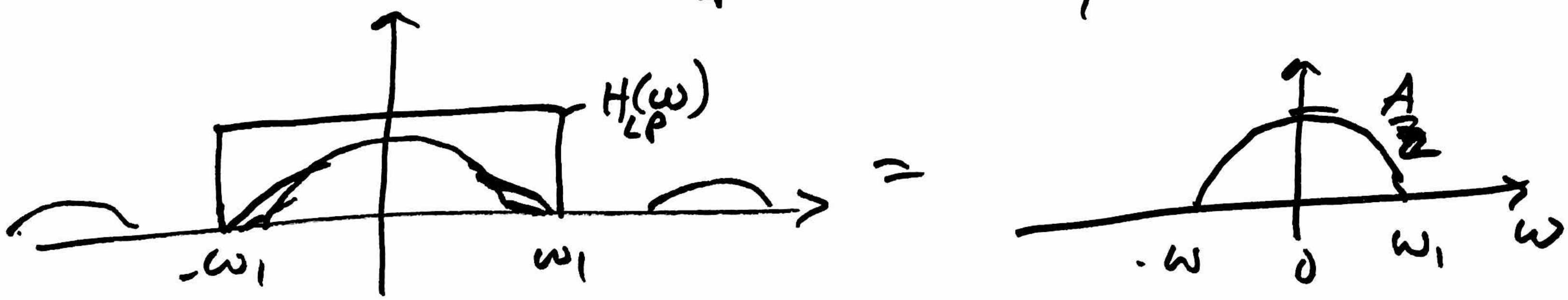


$R(\omega)$ contains the original $X(\omega)$ with spectral copies at $\pm 2\omega_0$.

Lowpass filter $R(\omega)$ to get $BX(\omega)$ back.

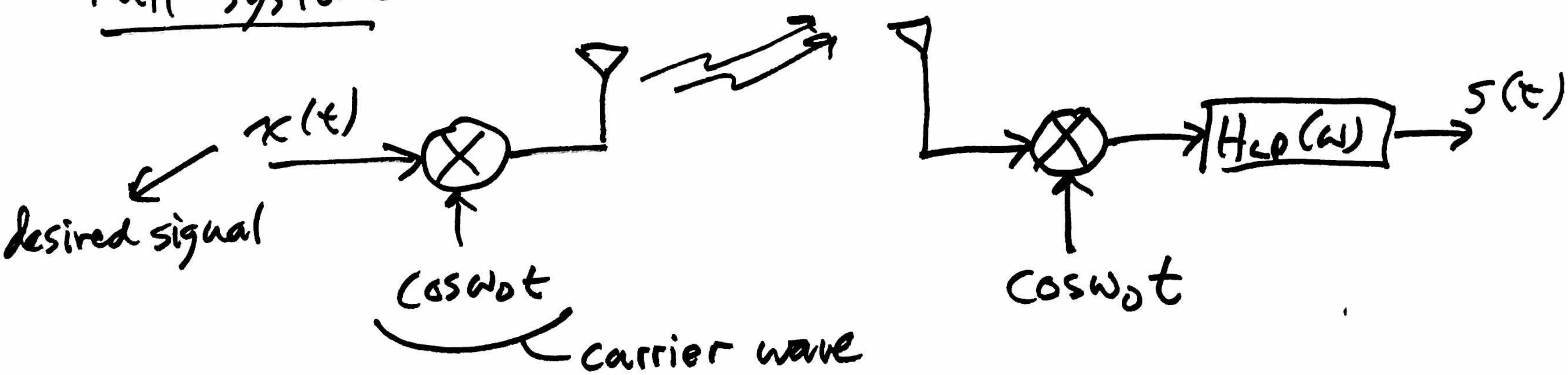


$$H_{LP}(\omega) = \begin{cases} 1, & |\omega| \le \omega_1 \\ 0, & \text{else} \end{cases}$$



$$s(t) = \tfrac{1}{2}x(t), \quad S(\omega) = \tfrac{1}{2}X(\omega)$$

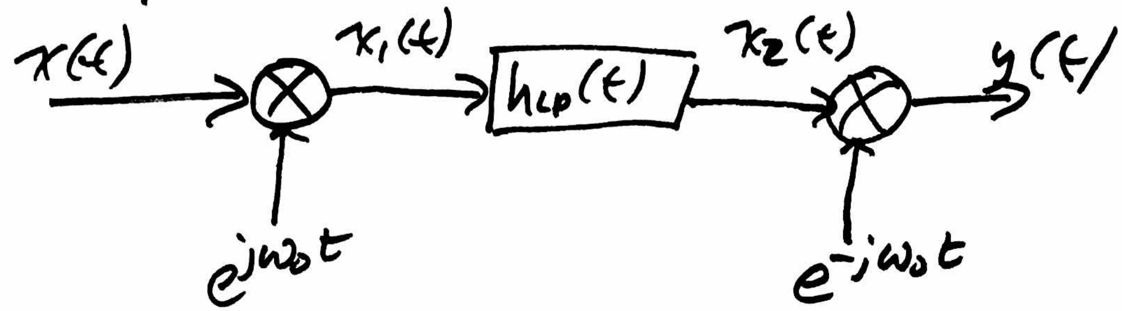
<u>Full system</u>



Key points: We can use a sinusoid to shift a spectrum.

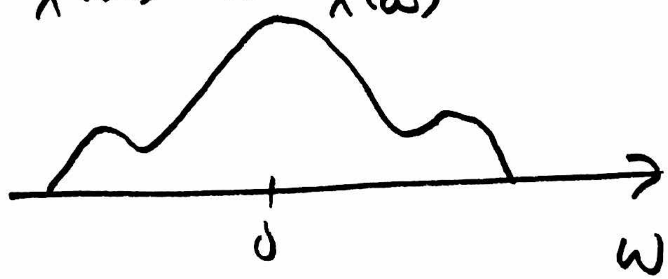
~~While~~ It can be simpler to find an appropriate filter while working in the frequency domain.

Ex frequency selective filtering

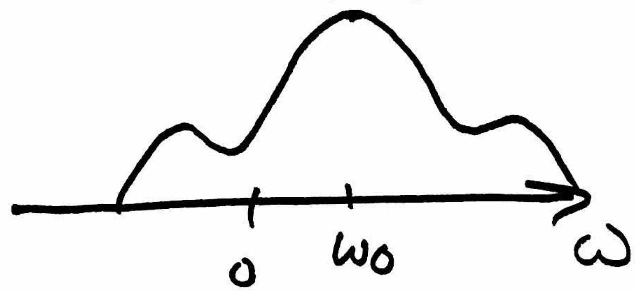


$$x_1(t) = x(t)e^{j\omega_0 t} \xleftrightarrow{CTFT} X(\omega-\omega_0)$$

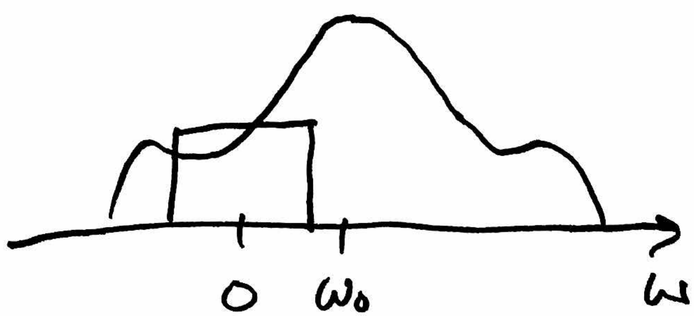
If $X(\omega)$ is $X(\omega)$



then $X_1(\omega)$

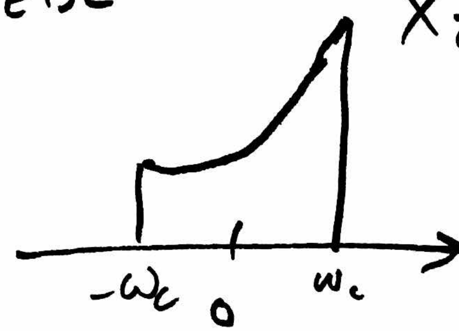


Then $X_2(\omega)$ if $H_{LP}(\omega) = \begin{cases} 1, & |\omega| < \omega_c \\ 0, & \text{else} \end{cases}$

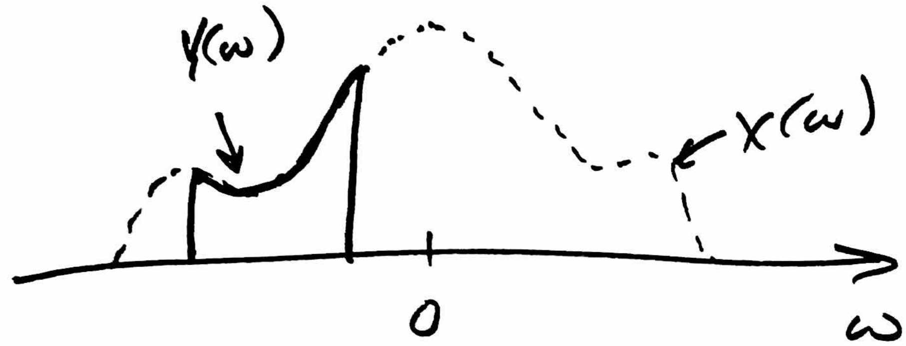


$=$

$X_2(\omega)$



$$y(t) = x_2(t)\, e^{-j\omega_0 t} \longleftrightarrow X(\omega+\omega_0)$$



We've implemented a bandpass filter using a lowpass filter. We can change the filter by modifying ω_0.