(15 pts) **1.** Compute the Fourier transform of the DT signal

$$x[n] = n^2u[n-2] - n^2u[n+2]$$

(Express your answer as a linear combination of sine and/or cosine functions.)

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} n^2 e^{-j\omega n} \left[u[n-2] - u[n+2]\right]$$

$$-\sum_{n=-2}^{2} n^2 e^{-j\omega n}$$

$$-(-2)^2 e^{-j\omega(-2)} - (-1)^2 e^{-j\omega(-1)} - (0) - (1)^2 e^{-j\omega(1)} - (2)^2 e^{-j\omega(2)}$$

$$-4e^{2j\omega} - 4e^{-2j\omega} - 1e^{-j\omega} - 1e^{-j\omega}$$

$$\boxed{-8\cos 2\omega - 2\cos\omega}$$

(15 pts) **2.** Show that the Fourier transform of the CT signal $x(t) = cos(\omega_0 t)$ is $\mathcal{X}(\omega) = \pi\delta(\omega + \omega_0) + \pi\delta(\omega - \omega_0)$.

$$x[n] = \frac{1}{2\pi}\int_{2\pi} \pi\,\delta(w + w_0)\, e^{jwn} + \pi\,\delta(w - w_0)\, e^{jwn}\, dw$$

$$= \frac{1}{2\cancel{\pi}} \cancel{\pi}\, e^{jwn}\Big|_{-w_0} + \frac{1}{2\cancel{\pi}}\cancel{\pi}\, e^{jwn}\Big|_{w_0}$$

$$= \frac{1}{2} e^{j(-w_0)n} + \frac{1}{2} e^{jw_0 n}$$

$$= \boxed{\cos(w_0 t)}$$

(15 pts) **3.** Given is a DT signal $x[n] = \frac{1}{g[n]^2}$ where $g[n]$ is a pure imaginary signal and an odd function of n.

a) Bob claims that the Fourier transform of $x[n]$ is $\mathcal{X}(\omega) = \frac{j}{cos\omega}$. Explain why Bob's answer is wrong.

If $x[n]$ is real and even, $X(\omega)$ must be real and even.
However, $\frac{j}{\cos\omega}$ is pure imaginary and even.

b) Alice says that the Fourier transform of $x[n]$ is $\mathcal{X}(\omega) = \frac{1}{sin\omega}$. Could Alice be right? Explain.

No, If $x[n]$ is real and even, $X(\omega)$ must be real and even,
However, $\frac{1}{\sin\omega}$ is real and odd.

c) Devin says that the Fourier transform of $x[n]$ is $\mathcal{X}(\omega) = \frac{1}{\omega^2}$. Could Devin be right? Explain.

No. Fourier transforms in DT must be periodic with period 2π.

4. A discrete-time LTI system has frequency response

$$H(e^{j\omega}) = \frac{2}{1 - \frac{3}{4}e^{-j\omega} + \frac{1}{8}e^{-2j\omega}}.$$

(15 pts) a) Derive a difference equation relating the input and the output of this system. (Use the properties of the Fourier transform listed in the table to justify your answer.)

$$Y(e^{j\omega}) = X(e^{j\omega}) \cdot H(e^{j\omega}) = \frac{2X(e^{j\omega})}{1 - \frac{3}{4}e^{-j\omega} + \frac{1}{8}e^{-2j\omega}}$$

$$Y(e^{j\omega}) - \frac{3}{4}Y(e^{j\omega})e^{-j\omega} + \frac{1}{8}Y(e^{j\omega})e^{-2j\omega} = 2X(e^{j\omega})$$

by linearity and time shifting:

$$\boxed{y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = 2x[n]}$$

(10 pts) b) What is the Fourier transform of the output when the input is $x[n] = \left(\frac{1}{4}\right)^n u[n]$?

$$X(\omega) = \frac{1}{1-\left(\frac{1}{4}\right)e^{-j\omega}}$$

$$Y(\omega) = X(\omega) \cdot H(\omega) = \boxed{\frac{2}{\left[1-\left(\frac{1}{4}\right)e^{-j\omega}\right]\left[1-\frac{3}{4}e^{-j\omega}+\frac{1}{8}e^{-2j\omega}\right]}}$$

(15 pts) b) Find the unit impulse response of this system.

$$H(e^{jw}) = \frac{2}{(\frac{1}{8})(e^{-jw})^2 - (\frac{3}{4})(e^{-jw}) + 1} = \frac{16}{(e^{-jw})^2 - 6(e^{-jw}) + 8} = \frac{16}{(e^{-jw}-4)(e^{-jw}-2)}$$

(by partial fraction expansion)

$$= \frac{8}{e^{-jw}-4} + \frac{-8}{e^{-jw}-2} = \frac{-2}{1-\frac{1}{4}e^{-jw}} + \frac{4}{1-\frac{1}{2}e^{-jw}}$$

$$F^{-1}(H(e^{jw})) = h[n] = -2F^{-1}\left(\frac{1}{1-\frac{1}{4}e^{-jw}}\right) + 4F^{-1}\left(\frac{1}{1-\frac{1}{2}e^{-jw}}\right)$$

$$\boxed{h[n] = \left[-2\left(\tfrac{1}{4}\right)^n + 4\left(\tfrac{1}{2}\right)^n\right] u[n]}$$

(20 pts) **5.** Use the definition of the Fourier transform (*not* the properties listed in the table) to prove the following Fourier transform property.

$$x(at+b) \xrightarrow{F} \frac{e^{j\omega\frac{b}{a}}}{-a}\mathcal{X}\left(\frac{\omega}{a}\right) \text{ for any } a, b \text{ real numbers with } a < 0.$$

$$X(\omega) = \int_{-\infty}^{\infty} x(at+b)\, e^{-j\omega t}\, dt$$

(let $z = at+b$ and $dz = a\,dt$)

$$= \frac{1}{a}\int_{\infty}^{-\infty} x(z)\, e^{-j\omega\left(\frac{z-b}{a}\right)}\, dz$$

$a(-\infty)+b \Rightarrow \infty$ b/c $a<0$

$a(\infty)+b \Rightarrow -\infty$

$$-e^{j\omega\frac{b}{a}}\left(\frac{1}{a}\right)\int_{-\infty}^{\infty} x(z)\, e^{-j\omega\left(\frac{z}{a}\right)}\, dz = \boxed{\frac{e^{j\omega\frac{b}{a}}}{-a} X\left(\frac{\omega}{a}\right)}$$

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