Representing aperiodic CT signals with Continuous Time Fourier Transform (CTFT)

- Not going go through the derivation.

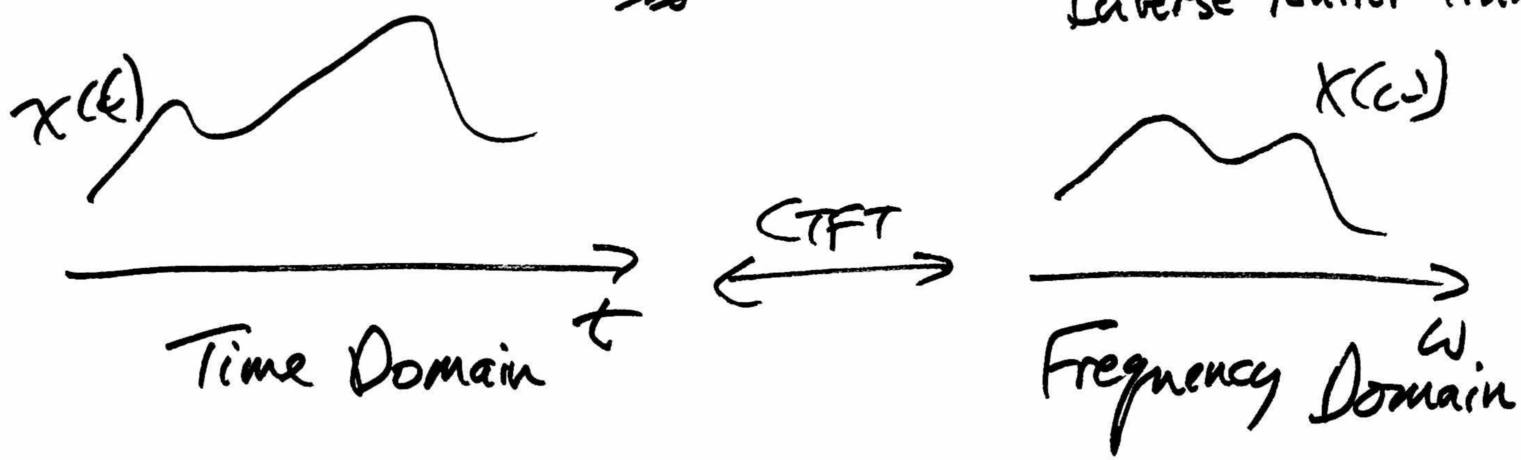
  End result:

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} \, dt \quad \text{Fourier transform}$$

spectrum

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} \, d\omega \quad \text{synthesis}$$

Inverse Fourier Transform



Recall from FS:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 t}$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega) e^{j\omega t}\, d\omega$$

$$\approx \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} X(k\Delta\omega) e^{jk\Delta\omega t}\,\Delta\omega$$

$$= \sum_{k=-\infty}^{\infty} \underbrace{\frac{\Delta\omega}{2\pi} X(k\Delta\omega)}_{\text{similiar to } a_k} e^{jk\Delta\omega t}$$

* Notation:

$X(\omega)$ will be used for CTFT & DTFT.

<u>Depends on context.</u>

CT: $X(j\omega)$ DT: $X(e^{j\omega})$

$$X(j\omega) = \frac{\sin(\frac{\omega}{2})}{\frac{\omega}{2}}$$

Ex

$$x(t) = e^{-at}u(t) \qquad a>0$$

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-j\omega t}\,dt$$

$$= \int_{0}^{\infty} e^{-at}e^{-j\omega t}\,dt = \int_{0}^{\infty} e^{-t(a+j\omega)}\,dt$$

$$= \frac{-1}{a+j\omega}\left[e^{-t(a+j\omega)}\right]_0^{\infty}$$

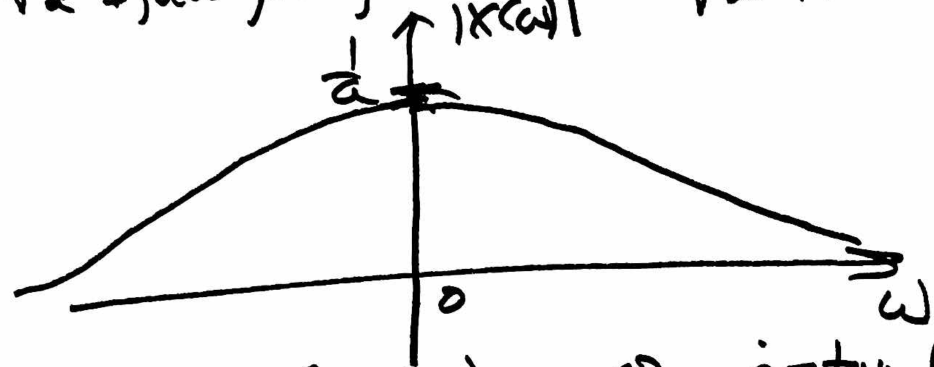
$$= -\frac{1}{a+j\omega}\left[\overset{0}{\cancel{e^{-\infty\cdot a}}}\,e^{-\infty\cdot j\omega} - \overset{1}{\cancel{e^{0}}}\right]$$

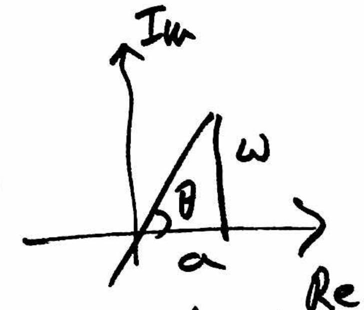
$$= \frac{1}{a+j\omega} = X(\omega)$$

Plot $|X(\omega)|$ & $\angle X(\omega)$ for $X(\omega) = \frac{1}{a+j\omega}$

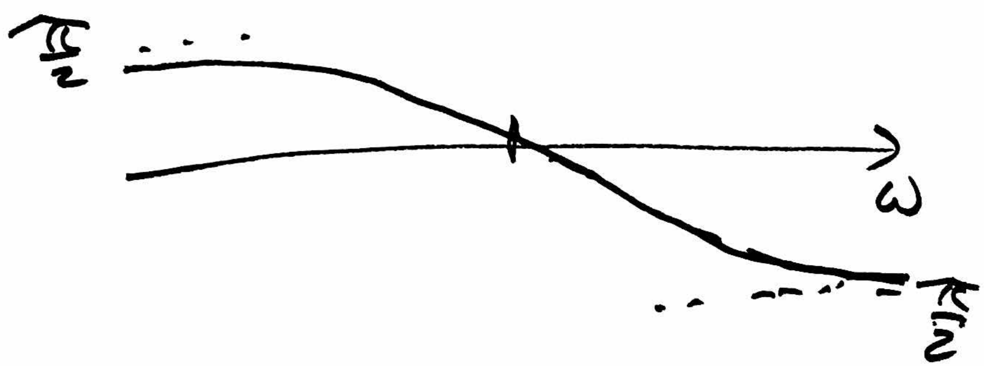
$$|X(\omega)| = \sqrt{\frac{1}{a+j\omega}\cdot\frac{1}{a-j\omega}} \qquad |C| = \sqrt{C\cdot C^*}$$

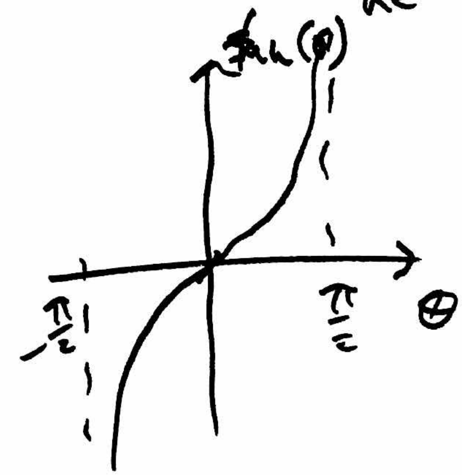
$$= \sqrt{\frac{1}{a^2+ja\omega-ja\omega-j^2\omega^2}} = \frac{1}{\sqrt{a^2+\omega^2}}$$





$$\angle X(\omega) = \angle 1 - \angle(a+j\omega) = 0 - \arctan\left(\frac{\omega}{a}\right)$$





Ex $x(t)=\delta(t)$

$$X(\omega)=\int_{-\infty}^{\infty}\delta(t)e^{-j\omega t}\,dt=\int_{-\infty}^{\infty}\delta(t)\cancel{e^{-j\omega 0}}^{\,1}\,dt$$

$$=1$$

ICTFT?

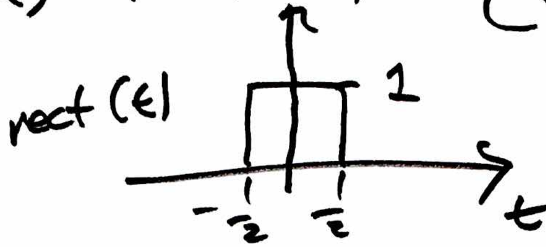
$X(\omega)=1$ $\quad x(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}X(\omega)e^{j\omega t}\,d\omega$ $\quad \frac{1}{2\pi}\cdot\frac{1}{jt}e^{j\omega t}\Big|_{-\infty}^{\infty}$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{j\omega t}\,d\omega=\frac{1}{2\pi}\,\frac{1}{jt}\left[e^{j\omega t}\right]_{-\infty}^{\infty}$$

does not immediately lead to the delta fn.

Ex

$$x(t) = \mathrm{rect}(t) = \begin{cases} 1 & |t| < \frac{1}{2} \\ 0 & \text{else} \end{cases}$$



unit pulse/rectangle

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt = \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-j\omega t} dt$$

$$= \frac{-1}{j\omega} \left[e^{-j\omega t} \right]_{-\frac{1}{2}}^{\frac{1}{2}} = -\frac{1}{j\omega} \cdot \left[e^{-j\frac{\omega}{2}} - e^{j\frac{\omega}{2}} \right]$$

$$= \frac{2}{\omega} \sin\left(\frac{\omega}{2}\right) = \frac{\sin\left(\frac{\omega}{2}\right)}{\frac{\omega}{2}} = \mathrm{sinc}\left(\frac{\omega}{2\pi}\right)$$

$$\mathrm{sinc}(t) = \frac{\sin(\pi t)}{\pi t}$$

$\frac{\omega}{2} = \pi t$

$t = \frac{\omega}{2\pi}$

Plotting sinc

$$\mathrm{sinc}(t) = \frac{\sin(\pi t)}{\pi t}$$

$$\mathrm{sinc}(t)\Big|_{t=0} = 1$$

$$\mathrm{sinc}(0) = \frac{\sin(0)}{0} = \frac{0}{0}$$

Use L Hopital's to get sinc(0) or

$\sin(\theta) \approx \theta$ for small θ

$$\frac{\sin(\pi t)}{\pi t} \approx \frac{\pi t}{\pi t} = 1 \text{ for small } t$$