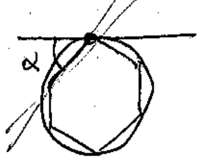
31 January 2012

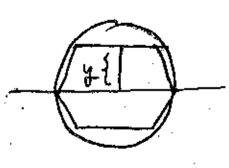
Recall:

Model A



versus

model B



angle $\alpha \rightarrow p(\text{secant} > \text{radius}) \leftarrow$ y-int.

$\frac{2}{3}$ = $\quad$ = $\frac{\sqrt{3}}{2}$.

model A: parametrized by <u>angles</u>

model B: parametrized by <u>y-int.</u>

thinking of calculating probability as ~~an~~ integrations

$$I = \int_{x=x_1}^{x=x_2} f(x)\,dx.$$

if $x = g(y)$ insted, then

$$I = \int_{g(y)=x_1}^{g(y)=x_2} f(g(y))\,d(g(y)) = \int f(g(y)) \cdot g'(y)\,dy.$$ ← chain rule

For model A, $f(\alpha) = \begin{cases} 0 & \text{if secant is incribed outside of hexagon} \\ 1 & \text{if not} \end{cases}$

model B, $f(\text{y-int}) = \begin{cases} 0 \\ 1 \end{cases}$ if y-int ~~grea~~ outside of hexagon.

We we frame the question, "all secant lines are equally distributed",

~~for model A~~ it may carry a different meaning for model A and B.

→ that is, what is equally distilbuted in terms of α may be a different "equally distributed" in terms of y-intercept.

If integration in parametrized by a diffrent variable, we cannot expect a linear mapping (they are two diff int in the first place...?)

<u>Random Variable & Expected Value</u>

<u>Def</u> S = sample space

X is a random variable on S.

If X is a function

$$X: S \rightarrow \mathbb{R}$$

e.g. S = collection of all possible outcome of two role of die.

$S = \{(1,1),(1,2), \ldots, (6,6)\}$ (21 possibilities total.), NOTE: $(1,2) \Leftrightarrow (2,1)$

Observe:

Not all outcomes have the same probability

$$p((i,i)) = \frac{1}{36}$$

$$p((i,j)) = \frac{1}{36} + \frac{1}{36} = \frac{1}{18}, \quad i \neq j.$$

Let X be the function $X((i,j)) = i + j \leftarrow \mathbb{R}$. so $X: S \to \mathbb{R}$

$Y((i,j)) = i \cdot j \leftarrow \mathbb{R}$ $\quad Y: S \to \mathbb{R}$

Def: The expected value $E(x)$ of a random variable is, by definition, the following some:

$$\sum_{s \in S} X(s) \cdot p(s) \qquad \text{(the weighted average of all possible outcomes)}$$

For X: (~~from~~ of previous example)

$$E(X) = \sum X(s) \cdot p(s)$$

$$= \sum_{i=1}^{6} X((i,i))\, p((i,i)) + \sum_{i=1}^{5}\sum_{j=1+i}^{6} X((i,j))\, p((i,j))$$

2+4+6+8+10+12

$$= \sum_{i=1}^{6} 2i \cdot \frac{1}{36} + \sum_{i=1}^{5}\sum_{j=1+i}^{6} (i+j) \cdot \frac{2}{36}$$

$$= \frac{21}{18} + \frac{1}{18}\sum_{i=1}^{5} i(6-i) + \frac{1}{18}\sum i(6-1)$$??? I can't read my handwriting...

magic

$$= \frac{7}{6} + \frac{1}{18}(35) + \frac{70}{18}$$

$$= \frac{21+35+70}{} = 7$$

Theorem

If X and Y are random variables on the same sample space S, then $E(X+Y) = E(Y) + E(X)$

(note: 1 roll $E(x) = 3.5$: two rolls $\therefore$ ~~$2E(x+x)$~~ $= E(2x) = (3.5)2 = 7.0$

Also $E(\lambda x) = \lambda E(X)$ if $\lambda \in \mathbb{R}$

~~For Y~~

~~$E(Y) = \sum_{s \in S} Y(s)\, p(s)$~~

For random variables ~~x~~ X and Y in the same sample space, we can only hope that $E(XY) = E(X)\,E(Y)$.

e.g.

Experiment: Rolling a Die

Let $Y(s) = s^2$

$X(s) = s$; does $E(X^2) = E(Y)$?

$$E(Y) = \frac{1}{6} + \frac{4}{6} + \frac{9}{6} + \frac{16}{6} + \dots = \frac{91}{6}.$$

$$E(X) = \frac{1}{6} + \frac{2}{6} + \dots + \frac{6}{6} = \frac{21}{6}$$

$$E(X^2) \neq E(X) \cdot E(X)$$

Q: Let S = {collection of all permutations of n elements.

The size of the sample space $|S| = n!$

Agree that

$$p(\sigma) = \frac{1}{n!} \quad \forall \sigma \in S$$

and

a <u>fixed point on σ</u> is a number that satisfies

$$\sigma(i) = i$$

We ~~not~~ define a random variable X

$$X : S \Rightarrow \mathbb{R}$$

$$X(\sigma) = \# \text{ of fixed points}$$

E.G.: If $n=3$

$$S = \{(1,2,3), (1,3,2), (2,1,3), (2,3,1), (3,1,2), (3,2,1)\}$$

then, $X = \{3, 1, 1, 0, 0, 1\}$

and $E(X) = \frac{1}{6}(3+1+1+0+0+1) = \underline{\underline{1}}$

$\Rightarrow$ we expect 1 ~~correct placement~~ fixed point..

if $n=2$

$$S = \{(1,2),(2,1)\}$$

$$X = \{2, \quad 0\}$$

$$E(X) = \frac{1}{2}(2+0) = 1 \checkmark$$

$n=1$

$$S = \{1\}$$

$$X = \{1\}$$

$$E(X) = 1(1) = 1 \quad \checkmark$$

Generally;

Let $x_i(\sigma) = \begin{cases} 1 & \text{if } \sigma_i = i \\ 0 & \text{if not} \end{cases}$

then, $X = \underset{\substack{\downarrow \\ \text{is } \sigma_1 \text{ fixedpt,}}}{x_1} + \underset{\substack{\downarrow \\ \text{is } \sigma_i \text{ fixed pt?}}}{x_2} + x_3 + \dots$ ...

then, $E(X) = \sum_{i=1}^{n} E(X_i)$

what is $E(x_i)$

$$E(x_i) = \frac{(n-1)! \leftarrow 1 \text{ fixed}}{n! \leftarrow \text{overall}} = \frac{1}{n}$$

?

since $E(x_i)$ is independent of i,

$$E(X) = \frac{1}{n} + \frac{1}{n} + \dots \frac{1}{n} = 1 \quad \checkmark$$

An inversion of σ is a pair $i<j$ with $\sigma(i) > \sigma(j)$

Let $Y(\sigma) = \#$ of inversions of σ.

what would be $E(Y)$?

for $n=3$

$$Y = 0, 1, 1, 2, 2, 3$$

$$E(Y) = \frac{1}{6}(0+1+1+2+2+3) = \frac{9}{6} = \frac{3}{2} \quad \dots$$ (to be continued)