EE 544 **Exam 1** **10/9/09**

Name: Solution

General Instructions:

- Write your name on every page of the exam.
- Do not write on the backs of the pages. If you need more paper, it will be provided to you upon request.
- The exam is closed book and closed notes, except that you may use one 8.5 by 11 inch crib sheet.
- Calculators are allowed.
- Your work must be explained to receive full credit.
- Point values for each problem are as indicated. The exam totals 100 points.
- All plots must be carefully drawn with axes labeled.

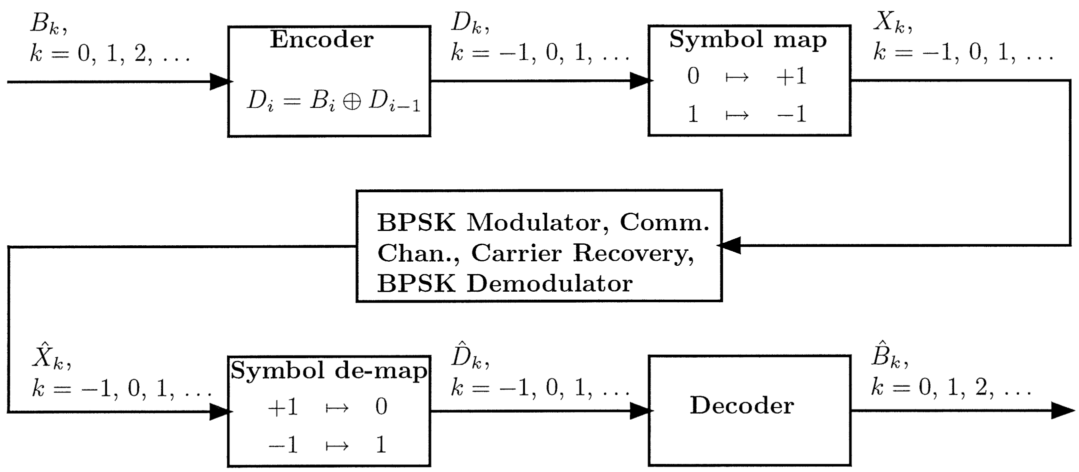
Do not open the exam until you are told to begin.

Description and Block Diagram for Problems 1 and 2

The block diagram shown is a possible implementation of a differentially encoded communication system.

The encoder accepts a binary (i.e., 0 or 1) string B_k at the input and produces a binary string D_k at the output. The output string contains an extra digit D_{-1}, which is set as an initial condition of the encoder. The symbol "$\oplus$" in the encoder denotes modulo 2 binary addition (i.e., exclusive or).

The decoder takes a binary string input $\hat{D}_k$ and produces a binary string output $\hat{B}_k$ containing one fewer digit than the input. The digits are lined up so that ideally $\hat{B}_k = B_k$ for $k = 0, 1, 2, \ldots$.



Problem 1. [25 pts. total] Consider the differentially encoded BPSK system described on the previous page focussing on the Encoder and Decoder blocks.

(a) [4 pts.] The table below gives an example input string B_k. Assuming that the encoder initial condition is $D_{-1} = 0$ as shown, fill in the values for the encoded bits D_k in the indicated row of the table.

	Time index k										
	-1	0	1	2	3	4	5	6	7	8	9
B_k	–	0	1	1	0	1	0	1	0	0	1
D_k	0	0	1	0	0	1	1	0	0	0	1
$\hat{D}_k$	0	0	1	0	0	1	1	0	0	0	1
$\hat{B}_k$	–	0	1	1	0	1	0	1	0	0	1

$D_k = B_k \oplus D_{k-1}$

(b) [7 pts.] The bits D_k starting from $k = -1$ are mapped to symbols, sent through the channel, and then de-mapped to produce the estimated bit sequence $\hat{D}_k$. Assuming that no bit errors occur in transmission, fill in the row in the above table corresponding to $\hat{D}_k$.

Then give a mathmatical formula for the decoder and write down the estimated bit sequence $\hat{B}_k$ in the above table.

$$\boxed{\hat{B}_k = \hat{D}_k \oplus \hat{D}_{k-1}} = D_k \oplus D_{k-1} \quad \text{if no channel errors}$$

$$= (B_k \oplus D_{k-1}) \oplus D_{k-1} \quad \text{substituting encoder formula.}$$

$$= B_k \oplus (D_{k-1} \oplus D_{k-1})$$

$$= B_k \oplus 0$$

$$= B_k \longrightarrow \text{In absense of channel errors the bit stream is reproduced.}$$

Problem 1. (cont'd.)

(c) [7 pts.] Repeat part (b) for the table shown below but now assume that the channel is such that the bits $\hat{D}_k$ are the complements of the corresponding bits D_k. Use exactly the same decoder as you found in part (b). The table is repeated below. You will need to fill the D_k row in with the same values as found in part (a).

Explain why the result you find is important in BPSK systems, which use either the squaring loop or the Costas loop for carrier phase recovery.

	Time index k										
	-1	0	1	2	3	4	5	6	7	8	9
B_k	–	0	1	1	0	1	0	1	0	0	1
D_k	0	0	1	0	0	1	1	0	0	0	1
$\hat{D}_k$	1	1	0	1	1	0	0	1	1	1	0
$\hat{B}_k$	–	0	1	1	0	1	0	1	0	0	1

Channel complementing all bits in D_k is what happens when carrier recovery loop locks on with a phase offset of π.

Since $\hat{B}_k = B_k$ in this case we see that differential encoding is insensitive to phase offset

Problem 1. (cont'd.)

(d) [7 pts.] Repeat part (b) only now assuming that a single bit error is made in the channel at time index $k = 3$, i.e., $\hat{D}_k = D_k$ for $k \neq 3$ and $\hat{D}_3 \neq D_3$. The position of the bit error is indicated in the table below with a small box.

Comment on the result in light of what was found in the homework problem about the probability of error performance of differentially encoded BPSK in comparison to regular BPSK.

	Time index k										
	-1	0	1	2	3	4	5	6	7	8	9
B_k	–	0	1	1	0	1	0	1	0	0	1
D_k	0	0	1	0	0	1	1	0	0	0	1
$\hat{D}_k$	0	0	1	0	$\boxed{1}$	1	1	0	0	0	1
$\hat{B}_k$	–	0	1	1	1	0	0	1	0	0	1

A single channel error produces two errors in the decoded bit stream.

In HW saw that the bit error prob. of DBPSK was approx twice that of BPSK at high SNR. Above is an illustration of why this happens.

Problem 2. [35 pts. total] This problem concerns just the encoder of the DBPSK system given before. Suppose that the input bit string B_k is independent and identically distributed (i.i.d.) with

$$P(B_k = 1) = p \quad \text{and} \quad P(B_k = 0) = 1 - p.$$

(a) [15 pts.] Assuming that the encoder initial state is $D_{-1} = 0$ find the marginal probability distribution of the encoder output for all time k, i.e., find

$$q_k \stackrel{\text{def}}{=} P(D_k = 1)$$

for $k \geq 0$. *Hint:* Find a first order difference equation for q_k and solve it.

(b) [10 pts.] For general p, are the random variables $\{D_k : k \geq 0\}$ identically distributed? Are they statistically independent? You must prove or give a counter example.

(c) [10 pts.] For the special case of $p = 1/2$, are the random variables $\{D_k : k \geq 0\}$ identically distributed? Are they statistically independent? You must prove or give a counter example.

(a) $D_k = B_k \oplus D_{k-1}$. Because of the dependence on past history it makes sense to condition on the value of D_{k-1} in determining $P(D_k = 1)$

$$P(D_k=1) = P(D_k=1 \mid D_{k-1}=0)P(D_{k-1}=0) + P(D_k=1 \mid D_{k-1}=1)P(D_{k-1}=1)$$
$$= P(B_k=1)(1-q_{k-1}) + P(B_k=0)\,q_{k-1}$$

$$\Longrightarrow \quad q_k = p(1-q_{k-1}) + (1-p)q_{k-1} = p - pq_{k-1} + q_{k-1} - pq_{k-1}$$
$$= (1-2p)\,q_{k-1} + p$$

The initial condition for this difference equation is $q_{-1} = P(D_{-1}=1) = 0$ since start with prob. one in state where $D_{-1}=0$. Then also have $q_0 = p$.

To solve the diff. equation we need only carry it out for a few steps until a pattern emerges.

Let $a = 1-2p$ for short.

Problem 2. (cont'd.)

$$q_{-1} = 0$$

$$q_0 = p$$

$$q_1 = ap + p$$

$$q_2 = a(ap+p)+p = a^2p + ap + p$$

$$q_3 = a(a^2p+ap+p) + p = a^3p + a^2p + ap + p$$

Pattern is clear:

$$q_k = p \sum_{i=0}^{k} a^i$$

To get a nice closed form we should simplify the sum.

Note:

$$\sum_{i=0}^{\infty} a^i = \frac{1}{1-a}; \quad \sum_{i=k+1}^{\infty} a^i = a^{k+1} \sum_{i=0}^{\infty} a^i = \frac{a^{k+1}}{1-a}$$

$$\Longrightarrow \frac{1}{1-a} - \frac{a^{k+1}}{1-a} = \frac{1-a^{k+1}}{1-a} = \sum_{i=0}^{k} a^i$$

$$\therefore \quad q_k = p\,\frac{1-a^{k+1}}{1-a} = p\,\frac{1-(1-2p)^{k+1}}{1-(1-2p)}$$

$$= p\,\frac{1-(1-2p)^{k+1}}{2p} = \frac{1}{2}\left[1-(1-2p)^{k+1}\right]$$

(b) It is obvious the D_k are not identically distributed in general because the marginal probabilities $P(D_k = 1)$ depend on k.

For statistical independence start by looking at

$$P(D_k = 1 \mid D_{k-1} = 0) \quad \text{and} \quad P(D_k = 1 \mid D_{k-1} = 1)$$

Problem 2. (cont'd.)

If they are statistically indep then the conditional probabilities should not depend on the conditioning variable. But here

$$P(D_k = 1 \mid D_{k-1} = 0) = P(B_k = 1) = p \neq q_k$$

$$P(D_k = \mid D_{k-1} = 1) = P(B_k = 0) = 1 - p \neq q_k$$

$\therefore$ $\{D_k\}$ not an indep. seq.

(c) Special case $p = 1/2$

Then $q_k = 1/2 \ \forall k$. Now marginal distributions of the $\{D_k\}$ are identical.

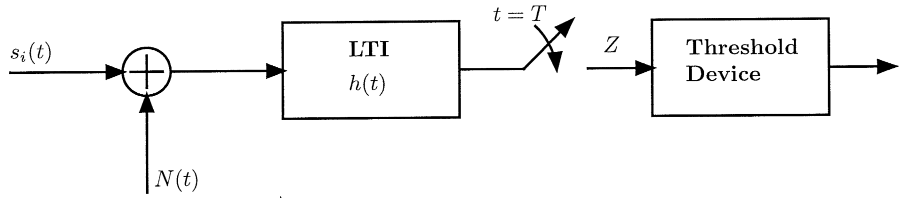
To show independence of the $\{D_k\}$ it is enough to show

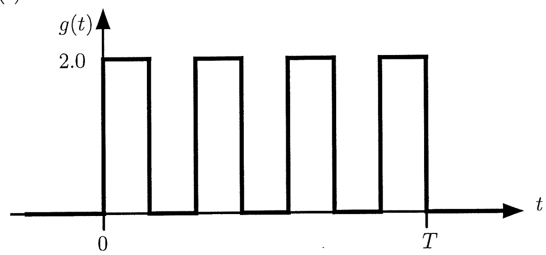
$$P(D_k = 1 \mid D_{k-1} = 0) = P(D_k = 1 \mid D_{k-1} = 1) = 1/2$$

since $\{D_k\}$ is a 1st order Markov process. Of course, the above holds just as it did in part (b).

Problem 3. [40 pts. total] Consider the binary baseband communication and basic pulse $g(t)$, which are shown in the figures below. The noise $N(t)$ is AWGN with power spectral density height equal to $N_0/2$. The signals under the two hypotheses are $s_0(t) = \alpha_0 g(t)$ and $s_1(t) = \alpha_1 g(t)$ where $\alpha_0 > \alpha_1$. In this problem the performance criterion is average probability of error (i.e., it is a Bayes problem) where the priors on the two signals are $\pi_0 = \pi_1 = 1/2$.

Let hypothesis H_i correspond to the transmission of signal s_i for $i = 0, 1$. Assume that the LTI filter $h(t)$ is chosen to be the matched filter for pulse shape $g(t)$ and sampling time $t = T$.





(a) [16 pts.] Find the probability density functions $f_i(z)$, $i = 0, 1$, for the decision statistic Z under the two hypotheses including writing the parameters of the pdfs in terms of α_0, α_1, N_0, and T.

(b) [10 pts.] Starting from the likelihood ratio test for this Bayesian hypothesis test show that an equivalent test is of the form

$$Z \begin{array}{c} > \\ \leq \end{array} \gamma \begin{array}{l} \text{decide } H_0 \\ \text{decide } H_1 \end{array}$$

and find the threshold γ for minimum average probability of error.

(c) [14 pts.] Find the average probability of error in terms of α_0, α_1, N_0, and T.

Problem 3. (cont'd.)

Under either hypothesis Z is Gaussian. Thus only need to find means and variance.

$$h(t) = g(T-t)$$

$$Z = \int [S_i(\tau) + N(\tau)] h(T-\tau)\, d\tau = \int_0^T [S_i(\tau) + N(\tau)]\, g(\tau)\, d\tau$$

$$= \underbrace{\alpha_i \int_0^T g^2(\tau)\, d\tau}_{\text{deterministic part}} + \underbrace{\int_0^T N(\tau) g(\tau)\, d\tau}_{\text{a zero mean Gaussian random variable}}$$

Re: Deterministic Part

$$\int_0^T g^2(t)\, dt = 4 \cdot \frac{4}{7} T = \frac{16}{7} T \implies E\{Z \mid H_i\} = \alpha_i \frac{16}{7} T$$

Re: Random Part

$$\mathrm{Var}\{Z\} = \int_0^T \int_0^T g(t) g(s)\, E\{N(t) N(s)\}\, dt\, ds = \int_0^T \int_0^T g(t) g(s) \frac{N_0}{2} \delta(t-s)\, dt\, ds$$

$$= \frac{N_0}{2} \int_0^T g^2(t)\, dt = \frac{N_0}{2} \frac{16}{7} T = \frac{8}{7} N_0 T$$

$\therefore$ Under H_i $\quad Z \sim N\left(\alpha_i \frac{16}{7} T, \frac{8}{7} N_0 T\right)$

(a) $$f_i(z) = \frac{1}{\sqrt{2\pi\, 8 N_0 T/7}}\, e^{-(z - \alpha_i 16T/7)^2 / (16 N_0 T/7)} \qquad i = 0, 1$$

(b) **Problem 3.** (cont'd.)

For simplicity let $\mu_i = \alpha_i 16T/7$, $\sigma^2 = 8N_0T/7$. The LRT for a Bayesian test is

$$L(z) = \frac{f_1(z)}{f_0(z)} \underset{<}{\geq} \tau = \frac{\pi_0}{\pi_1} = 1 \quad \begin{matrix}\cdots H_1 \\ \cdots H_0\end{matrix} \quad \text{for } \pi_0 = \pi_1 = 1/2$$

$$L(z) = e^{-(z-\mu_1)^2/2\sigma^2} \Big/ e^{-(z-\mu_0)^2/2\sigma^2}$$

$$= \exp\left\{-\frac{1}{2\sigma^2}\left[(z-\mu_1)^2 - (z-\mu_0)^2\right]\right\}$$

Take nat. log.

$$L(z) \underset{<}{\geq} \tau = 1 \iff \ln L(z) = -\frac{1}{2\sigma^2}\underbrace{\left[(z-\mu_1)^2 - (z-\mu_0)^2\right]}_{\begin{matrix} z^2 - 2\mu_1 z + \mu_1^2 \\ -z^2 + 2\mu_0 z - \mu_0^2 \end{matrix}} \underset{<}{\geq} 0$$

$$\iff -\frac{1}{2\sigma^2}\left[-2(\mu_1-\mu_0)z + \mu_1^2 - \mu_0^2\right] \underset{<}{\geq} 0 \quad \begin{matrix} H_1 \\ H_0 \end{matrix}$$

$$\iff \frac{\mu_1 - \mu_0}{\sigma^2}\left[z - \frac{\mu_1+\mu_0}{2}\right] \underset{<}{\geq} 0 \quad \begin{matrix} H_1 \\ H_0 \end{matrix}$$

$$\iff z - \frac{\mu_1+\mu_0}{2} \underset{\leq}{>} 0 \quad \begin{matrix} H_0 \\ H_1 \end{matrix} \qquad (\text{note: } \mu_1 < \mu_0).$$

Finally see the LRT test is equivalent to

$$z \underset{\leq}{>} \underbrace{\frac{\mu_1+\mu_0}{2}}_{\gamma} \quad \begin{matrix} \text{decide } H_0 \\ \text{" } H_1 \end{matrix}$$

From the Bayesian development we know this test gives min. prob. of error.

(c) **Problem 3.** (cont'd.)

$$P_{e,0} = P\left(Z \le \frac{\mu_1+\mu_0}{2} \,\Big|\, H_0\right) = P\left(\frac{Z-\mu_0}{\sigma} \le \frac{\mu_1-\mu_0}{2\sigma} \,\Big|\, H_0\right)$$

$$= \Phi\left(\frac{\mu_1-\mu_0}{2\sigma}\right) = Q\left(\frac{\mu_0-\mu_1}{2\sigma}\right)$$

$$P_{e,1} = P\left(Z > \frac{\mu_1+\mu_0}{2} \,\Big|\, H_1\right) = P\left(\frac{Z-\mu_1}{\sigma} > \frac{\mu_0-\mu_1}{2\sigma} \,\Big|\, H_1\right)$$

$$= Q\left(\frac{\mu_0-\mu_1}{2\sigma}\right)$$

$$\overline{P_e} = \pi_0 P_{e,0} + \pi_1 P_{e,1} = Q\left(\frac{\mu_0-\mu_1}{2\sigma}\right) \quad \text{since } \pi_0 = \pi_1 = 1/2$$

$$\text{SNR} = \frac{\mu_0-\mu_1}{2\sigma} = \frac{\frac{\alpha_0 16T}{7} - \frac{\alpha_1 16T}{7}}{2\sqrt{\frac{8N_0T}{7}}} = \frac{\frac{16T}{7}(\alpha_0-\alpha_1)}{2\sqrt{\frac{8N_0T}{7}}} = \frac{\frac{8T}{7}(\alpha_0-\alpha_1)}{\sqrt{8N_0T/7}}$$

$$= \sqrt{\frac{8^{\cancel{2}}T^{\cancel{2}}}{7^{\cancel{2}}}(\alpha_0-\alpha_1)^2 \Big/ \cancel{8}N_0\cancel{T}/\cancel{7}} = \sqrt{\frac{8}{7}(\alpha_0-\alpha_1)^2\frac{T}{N_0}}$$