

## e.g. 5) Integrator

$$y(t) = \int_{t-T_b}^{t+T_a} x(z) dz$$

: area under  $x(t)$   
from  $t-T_b$  to  $t+T_a$

,  $T_a, T_b > 0$

. Linear? Yes.

$$y(t) = \int_{t-T_b}^{t+T_a} (\underline{a_1 x_1(z) + a_2 x_2(z)}) dz$$

$$= a_1 \int_{-}^{t+T_a} x_1(z) dz + a_2 \int_{-}^{t+T_a} x_2(z) dz$$

$$= \underline{a_1 y_1(t) + a_2 y_2(t)}$$

(2)

TI? ✓ Yes input  $x(t - t_0)$

$$z(t) = \int_{t-T_b}^{t+T_a} x(z-t_0) dz$$

use change of variables :  $\lambda = z - t_0$

$$\rightarrow d\lambda = dz, z = \lambda + t_0$$

$$z(t) = \int_{t-t_0-T_b}^{t-t_0+T_a} x(\lambda) d\lambda$$

$$= \underline{y(t - t_0)}$$

∴ System is LTI

(3)

## e.g. 6) Difference Equation

$$y_1[n] = - \sum_{k=1}^N a_k y_1[n-k] + \sum_{k=0}^M b_k x_1[n+k] \dots \textcircled{1}$$

$$x_1[n] \rightarrow [s] \rightarrow y_1[n]$$

$$y_2[n] = - \sum_{k=1}^N a_k y_2[n-k] + \sum_{k=0}^M b_k x_2[n+k] \dots \textcircled{2}$$

$$x_2[n] \rightarrow [s] \rightarrow y_2[n]$$

Consider  $a_1\textcircled{1} + a_2\textcircled{2}$

$$a_1 y_1[n] + a_2 y_2[n] =$$

$$- a_1 \sum_{k=1}^N a_k y_1[n-k] - a_2 \sum_{k=1}^N a_k y_2[n-k]$$

$$+ a_1 \sum_{k=0}^M b_k x_1[n+k] + a_2 \sum_{k=0}^M b_k x_2[n+k]$$

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$$= - \sum_{k=1}^N a_k \{ d_1 y_1[n-k] + d_2 y_2[n-k] \}$$

$$+ \sum_{k=0}^M b_k \{ d_1 x_1[n-k] + d_2 x_2[n-k] \}$$

→ Linear? Yes

Proof invokes distributive property  
of multiplication,

TI?

$$y[n-n_0] = - \sum_{k=1}^N a_k y[n-n_0-k] + \sum_{k=0}^M b_k x[n-n_0-k]$$

... ③

If the system is TI,  $x[n-n_0] \rightarrow \boxed{S} \rightarrow y[n-n_0]$

such that this input-output system satisfies  
the difference eq.

$$y[n-n_0] = - \sum_{k=1}^N a_k y[n-k-n_0] + \sum_{k=0}^M b_k x[n-k-n_0]$$

... ④

Due to the commutativity of addition.

eq. ③ and ④ are the same

→ System is TI.

Comments :

- Difference eq. with constant coefficients are LTI systems.
- Difference eq. are used all the time in practice for a variety of applications.
- They can be used as freq. selective filters (lowpass, bandpass, highpass), notch filter, and many other functions.
- Since they are LTI, they are completely characterized by their impulse response.

$$\text{e.g. 1)} \quad y[n] = \sum_{k=n-n_0}^{n+n_0} x[k]$$

This is a difference eq. It's just non-recursive  
( $a_k=0$ ) and non-causal.

So it's LTI, but we'll prove it anyhow.

- Linear? Yes

$$\textcircled{1} \quad y_1[n] = \sum_{k=n-n_0}^{n+n_0} x_1[k] \quad \textcircled{2} \quad y_2[n] = \sum_{k=n-n_0}^{n+n_0} x_2[k]$$

Consider  $d_1\textcircled{1} + d_2\textcircled{2}$

$$\Rightarrow d_1 y_1[n] + d_2 y_2[n] = d_1 \underbrace{\sum_{k=n-n_0}^{n+n_0} x_1[k]}_{\text{blue line}} + d_2 \underbrace{\sum_{k=n-n_0}^{n+n_0} x_2[k]}_{\text{blue line}}$$

$$= \sum_{k=n-n_0}^{n+n_0} \{ d_1 x_1[k] + d_2 x_2[k] \}$$

TI? Yes

(note: since  $n_0$  is used in system equation.  
we'll  $\ell$  for time-shift in Input signal.)

$$x[n-\ell] \rightarrow [S] \rightarrow z[n]$$

"IS  $z[n] = y[n-\ell]$ ?"

$$z[n] = \sum_{k=n-n_0}^{n+n_0} x[k-\ell]$$

Change of variables :  $k' = k - \ell$

$$\rightarrow k = k' + \ell$$

$$z[n] = \sum_{k'=n-\ell-n_0}^{n+\ell+n_0} x[k'] \quad \leftarrow$$

$$\text{Recall: } y[n] = \sum_{k=n-n_0}^{n+n_0} x[k] \Rightarrow y[n-\ell] = \sum_{k=n-\ell-n_0}^{n-\ell+n_0} x[k]$$

$$z[n] = y[n-\ell]$$

\* system is stable.