$\boxed{\text{O+W 3.28 (c, d)}}$

Find the DTFS coeffs for

(c) $x[n]$ periodic of period 4 and

$$x[n] = 1 - \sin\left(\frac{\pi n}{4}\right) \qquad 0 \le n \le 3$$

Note that the periodic seq. $x[n]$ is defined by the given formula for $n = 0, 1, 2, 3$ and then periodically extended to have period $N = 4$. Thus the general formula for DTFS is

$$X_k = \frac{1}{4} \sum_{n=0}^{3} x[n]\, e^{-j2\pi kn/4} \qquad k = 0, 1, 2, 3$$

$$= \frac{1}{4}\left[\underbrace{1}_{=x[0]} + \underbrace{\frac{\sqrt{2}-1}{\sqrt{2}}}_{=x[1]} e^{-j\pi k/2} + \underbrace{\frac{\sqrt{2}-1}{\sqrt{2}}}_{=x[3]} e^{-j3\pi k/2}\right] \qquad k = 0, 1, 2, 3$$

$\left(\text{and } x[2] = 0\right)$

$$= \frac{1}{4}\left[1 + \frac{\sqrt{2}-1}{\sqrt{2}}(-j)^k + \frac{\sqrt{2}-1}{2}(j)^k\right]$$

$$= \frac{1}{4}\left[1 + \left(\frac{\sqrt{2}-1}{\sqrt{2}}\right)\left((-j)^k + (j)^k\right)\right]$$

$$= \begin{cases} \frac{3-\sqrt{2}}{4} & k = 0 \\ 1/4 & k = 1 \\ -\frac{1+\sqrt{2}}{4} & k = 2 \\ 1/4 & k = 3 \end{cases}$$

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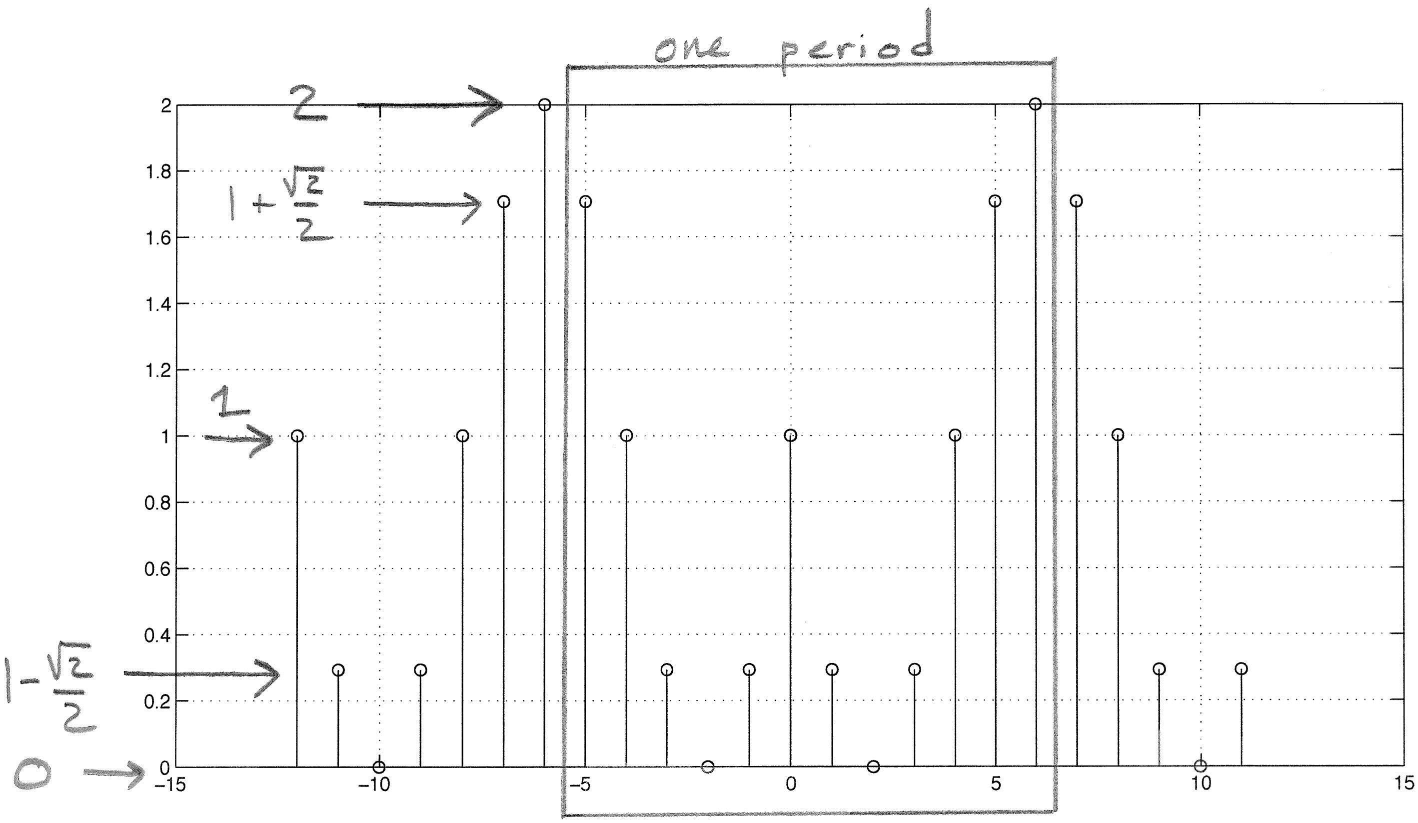
(d) $x[n]$ periodic with period $N = 12$ and

$$x[n] = 1 - \sin\left(\frac{\pi n}{4}\right) \qquad 0 \le n \le 11$$

Again we note that the signal $x[n]$ is created by periodically extending the values computed from the formula above. See the plot below:

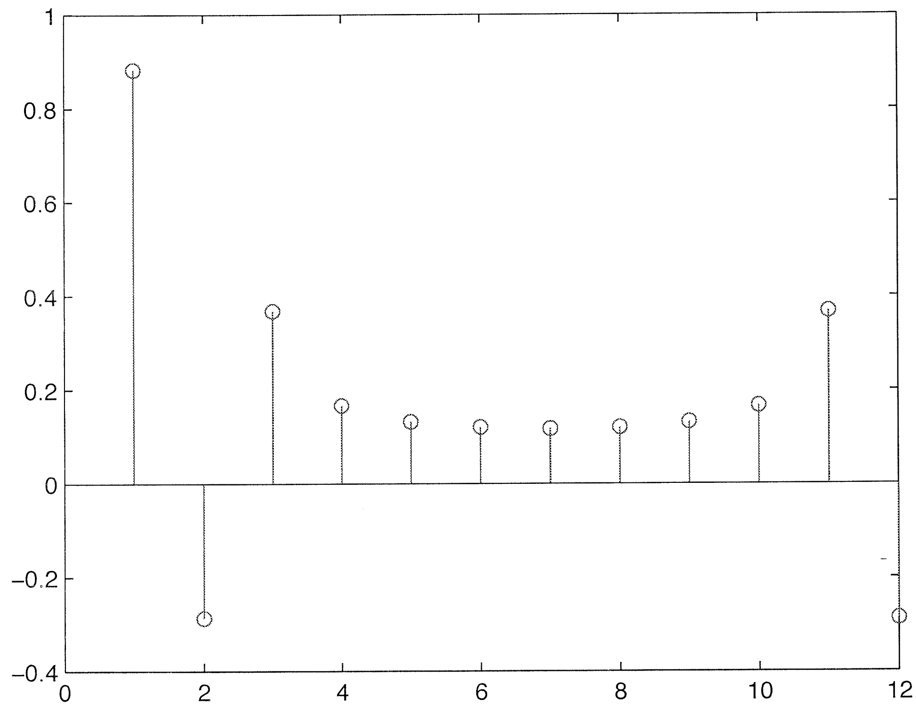


$$X_k = \frac{1}{12}\sum_{n=0}^{11} x[n]\, e^{-j2\pi kn/12}$$

$$= \frac{1}{12}\sum_{n=-5}^{6} x[n]\, e^{-j2\pi kn/12} \qquad \left(\begin{array}{l}\text{if use the correct}\\ \text{periodically extended}\\ \text{signal}\end{array}\right)$$

$$= \frac{1}{12}\Bigg[\left(1+\frac{\sqrt{2}}{2}\right)\left(e^{j5\pi k/6} + e^{-j5\pi k/6}\right) + 1\cdot\left(e^{j4\pi k/6} + 1 + e^{-j4\pi k/6}\right)$$
$$+ \left(1-\frac{\sqrt{2}}{2}\right)\left(e^{j3\pi k/6} + e^{j\pi k/6} + e^{-j\pi k/6} + e^{-j3\pi k/6}\right) + 2e^{-j6\pi k/6}\Bigg]$$

$$X_k = \frac{1}{12}\Big[(2+\sqrt{2})\cos\left(\frac{5\pi k}{6}\right) + 2\cos\left(\frac{2\pi k}{3}\right) + (2-\sqrt{2})\left(\cos\left(\frac{\pi k}{6}\right) + \cos\left(\frac{\pi k}{2}\right)\right) + 2\left(1+(-1)^k\right)\Big] \quad (*) \qquad k=0,1,\dots,11$$



Above computed directly from the formula (*). Could also compute using Matlab command fft or ifft (but be careful as O+W have a slightly different definition for DFT coeffs than does Mathworks).

$\frac{2}{10}$

ECE 301 Signals and Systems
Homework # 7 Solution

$\boxed{3.29}$ b) $a_k = \begin{cases} \sin(\frac{k\pi}{3}) & , 0 \le k \le 6 \\ 0 & , k = 7 . \end{cases}$

$$x[n] = \sum_{0}^{7} a_k e^{-jk\omega_0 n}$$
$$= \frac{\sqrt{3}}{2} e^{-j\frac{\pi}{4}n} + \frac{\sqrt{3}}{2} e^{-j\frac{\pi}{2}n} - \frac{\sqrt{3}}{2} e^{-jn\pi} - \frac{\sqrt{3}}{2} e^{-j\frac{5\pi}{4}n}.$$

d) $x[n] = \sum_{-3}^{4} a_k e^{-jk\omega_0 n}$

$$= 2 + 1 \cdot (e^{-j\frac{\pi}{4}n} + e^{j\frac{\pi}{4}n}) + \frac{1}{2}(e^{-j\frac{\pi}{2}n} + e^{j\frac{\pi}{2}n}) + \frac{1}{4}(e^{-j\frac{3\pi}{4}n} + e^{j\frac{3\pi}{4}n})$$

$$= 2 + 2\cos(\frac{\pi}{4}n) + \cos(\frac{\pi}{2}n) + \frac{1}{2}\cos(\frac{3\pi}{4}n).$$

$\boxed{\text{O+W 3.36 (b)}}$

Causal LTI system where input and output are related by a difference equation:

$$x[n] \longrightarrow \boxed{\begin{matrix}\text{Causal}\\ \text{LTI}\end{matrix}} \xrightarrow{\;y[n]\;} \quad ; \quad y[n] - \frac{1}{4}y[n-1] = x[n] \text{ for } n \in \mathbb{Z}$$

Find the Fourier series rep. for $y[n]$ for input

(b) $x[n] = \cos\left(\frac{\pi}{4}n\right) + 2\cos\left(\frac{\pi}{2}n\right)$

First lets review a few background points about LTI systems. If a general complex exponential signal in discrete time ie $x[n] = z^n \;\; \forall n \in \mathbb{Z}$ is applied to the input of an LTI system with impulse response $h[n]$ then

$$y[n] = H(z)\, z^n \quad \forall n \in \mathbb{Z}$$

where the system function $H(z)$ is formally defined to be

$$H(z) = \sum_{k=-\infty}^{\infty} h[k]\, z^{-k}$$

Of course for the above to make sense we need the sum in the definition of system function to converge. This requirement will constrain the allowable complex numbers $z \in \mathbb{C}$.

In the case where the input signal $x[n]$ is N-periodic then it may be represented by a DTFS expansion of the form

$$x[n] = \sum_{k=0}^{N-1} X_k e^{+j2\pi kn/N} \qquad (*)$$

so the elementary complex exponential signals of interest are complex sinusoids of the form

$$e^{j\omega n} \qquad n \in \mathbb{Z}$$

where the frequencies are of the form $\omega = \frac{2\pi}{N} k$; $k = 0, 1, 2 \ldots N-1$. Then the system function of interest is

$$H(z)\Big|_{z=e^{j\omega}} = H(e^{j\omega}) = \sum_{\ell=-\infty}^{\infty} h[\ell] e^{-j\omega \ell}$$

assuming the sum converges.

For the given difference equation there are two ways we could go for finding $H(e^{j\omega})$. The first would be to solve for the impulse response. Using previously covered methods can show:

$$h[n] = \left(\tfrac{1}{4}\right)^n u[n]$$

$$\Rightarrow H(e^{j\omega}) = \sum_{\ell=0}^{\infty} \left(\tfrac{1}{4}\right)^{\ell} e^{-j\omega \ell} = \sum_{\ell=0}^{\infty} \left(\tfrac{1}{4} e^{-j\omega}\right)^{\ell}$$

$$= \frac{1}{1 - \frac{1}{4} e^{-j\omega}}$$

The second approach is to assume an input of the form $x[n] = e^{j\omega n}$ and apply it to the LTI system using the fact that $y[n] = H(e^{j\omega}) e^{j\omega n}$

Then

$$y[n] - \frac{1}{4} y[n-1] = x[n]$$

$$H(e^{j\omega}) e^{j\omega n} - \frac{1}{4} H(e^{j\omega}) e^{j\omega(n-1)} = e^{j\omega n}$$

Cancelling the $e^{j\omega n}$ term:

$$H(e^{j\omega}) \left[1 - \frac{1}{4} e^{-j\omega} \right] = 1$$

$$\Longrightarrow \quad H(e^{j\omega}) = \frac{1}{1 - \frac{1}{4} e^{-j\omega}}$$

Now if an input of the form $e^{j2\pi kn/N}$ $n \in \mathbb{Z}$ is applied to the system the output will be

$$H\left(e^{j2\pi k/N}\right) e^{j2\pi kn/N} \qquad n \in \mathbb{Z}$$

$$= \frac{1}{1 - \frac{1}{4} e^{-j2\pi k/N}} e^{j2\pi kn/N}$$

Then linearity says that if an $x[n]$ of the form ($*$) is applied to the system the output will be

$$y[n] = \sum_{k=0}^{N-1} \frac{X_k}{1-\frac{1}{4}e^{-j2\pi k/N}} e^{j2\pi kn/N}$$

In otherwords, $y[n]$ is also N periodic and these are its DTFS coefficients Y_k $k=0,1,\ldots,N-1$

<u>Back to (b)</u> Need the DTFS for

$$x[n] = \cos\left(\frac{\pi}{4}n\right) + 2\cos\left(\frac{\pi}{2}n\right)$$

$$= \frac{1}{2}e^{-j\pi n/4} + \frac{1}{2}e^{+j\pi n/4} + e^{-j\pi n/2} + e^{+j\pi n/2}$$

$$= \frac{1}{2}e^{-j2\pi n/8} + \frac{1}{2}e^{+j2\pi n/8} + e^{-j4\pi n/8} + e^{+j4\pi n/8}$$

$\Longrightarrow$ Therefore periodic of period $N=8$ and DTFS coefficients

$$X_{-1} = X_1 = \frac{1}{2} \qquad X_{-2} = X_2 = 1$$

Thus the DTFS for $y[n]$ are

$$Y_{-1} = \frac{1/2}{1 - \frac{1}{4} e^{j\pi/4}} \qquad Y_1 = \frac{1/2}{1 - \frac{1}{4} e^{-j\pi/4}}$$

$$Y_{-2} = \frac{1}{1 - \frac{1}{4} e^{j\pi/2}} \qquad Y_2 = \frac{1}{1 - \frac{1}{4} e^{-j\pi/2}}$$

These could be simplified by using:

$$e^{j\pi/4} = \frac{\sqrt{2}}{2}(1+j) ; \quad e^{-j\pi/4} = \frac{\sqrt{2}}{2}(1-j)$$

$$e^{j\pi/2} = j ; \quad e^{-j\pi/2} = -j$$

but we can stop here and feel plenty happy.

O+W 3.48 (b, d, h)

$$x[n] = x[n+N] \quad \forall n \in \mathbb{Z}$$

$$x[n] = \sum_{k=0}^{N-1} a_k e^{jk(2\pi/N)n} \quad (*)$$

Find the DTFS of the signals defined below in terms of the a_k defined in $(*)$.

b) $y[n] \triangleq x[n] - x[n-1]$

Directly from $(*)$ have

$$y[n] = \sum_{k=0}^{N-1} a_k e^{jk(2\pi/N)n} - \sum_{k=0}^{N-1} a_k e^{jk(2\pi/N)(n-1)}$$

$$= \sum_{k=0}^{N-1} a_k \left(1 - e^{-j2\pi k/N}\right) e^{jk(2\pi/N)n}$$

$$\therefore \; y[n] \longleftrightarrow a_k\left(1 - e^{-j2\pi k/N}\right)$$

d) $w[n] = x[n] + x[n + N/2]$ assume N is even.

Note that $w[n]$ is periodic of period $N/2$ (and is also periodic of period N). So we could find DTFS for $w[n]$ assuming either period.

From same technique used in (b):

$$w[n] = \sum_{k=0}^{N-1} a_k\left(1 + e^{j2\pi(N/2)k/N}\right) e^{jk(2\pi/N)n}$$

Thus as an N-periodic sequence

$$w[n] \longleftrightarrow a_k\left(1+e^{j\pi k}\right) = a_k\left(1+(-1)^k\right)$$

$$= \begin{cases} 2a_k & k \text{ even} \\ 0 & k \text{ odd} \end{cases}$$

By some elementary (but tedious) manipulations of

$$w[n] = \sum_{k=0}^{N-1} a_k\left[1+(-1)^k\right] e^{j2\pi kn/N}$$

$$= \sum_{l=0}^{\frac{N}{2}-1} 2a_{2l}\, e^{j2\pi ln/(N/2)}$$

$\therefore$ as an $\frac{N}{2}$-periodic sequence

$$w[n] \longleftrightarrow 2a_{2k}$$

(Derivation on following pages).

Try to modify (**) $\sum_{k=0}^{N-1} a_k (1 + (-1)^k) e^{j2\pi kn/N}$ to make it look like a DTFS for an $\frac{N}{2}$ - periodic signal.

Two cases to consider: $\frac{N}{2}$ even and $\frac{N}{2}$ odd

Case : $\frac{N}{2}$ even

$$(**) = 2a_0 + 2a_2 e^{j2\pi 2n/N} + \cdots + 2a_{\frac{N}{2}-2} e^{j2\pi(\frac{N}{2}-2)n/N}$$

$$+ 2a_{\frac{N}{2}} e^{j2\pi(\frac{N}{2})n/N} + \cdots + 2a_{N-2} e^{j2\pi(N-2)n/N}$$

There are $\frac{N}{4} - 1$ (potentially) nonzero terms on each line above. Lets define

$$N = 4L \implies \frac{N}{2} = 2L, \ \frac{N}{4} = L$$

$$(**) = 2a_0 + 2a_2 e^{j2\pi n/2L} + \cdots + 2a_{2(L-1)} e^{j2\pi(L-1)n/2L}$$

$$+ 2a_{2L} e^{j2\pi Ln/2L} + 2a_{2(L+1)} e^{j2\pi(L+1)n/2L} + \cdots + 2a_{2(2L-1)} e^{j2\pi(2L-1)n/2L}$$

$$= \sum_{\ell=0}^{2L-1} 2a_{2\ell} e^{j2\pi \ell n/2L} = \sum_{\ell=0}^{\frac{N}{2}-1} 2a_{2\ell} e^{j2\pi \ell n/(N/2)}$$

$\therefore$ as an $\frac{N}{2}$ periodic sequence

$$w[n] \longleftrightarrow 2a_{2k}$$

Case: $\frac{N}{2}$ odd

$$(**) = 2a_0 + 2a_2 e^{j2\pi 2n/N} + \dots + 2a_{\frac{N}{2}-1} e^{j2\pi(\frac{N}{2}-1)n/N}$$

$$+ 2a_{\frac{N}{2}+1} e^{j2\pi(\frac{N}{2}+1)n/N} + \dots + 2a_{N-2} e^{j2\pi(N-2)n/N}$$

But now there are potentially an odd number of nonzero terms in the above sum. Let

$N = 2K$ where K is odd

$$(**) = 2a_0 + 2a_2 e^{j2\pi n/K} + \dots + 2a_{K-1} e^{j2\pi(K-1)n/N}$$

$$+ 2a_{K+1} e^{j2\pi(K+1)n/N} + \dots + 2a_{2(K-1)} e^{j2\pi(K-1)n/K}$$

Of course if K is odd then both $K-1$ and $K+1$ are even so

$$(**) = 2a_0 + 2a_2 e^{j2\pi n/K} + \dots + 2a_{2(\frac{K-1}{2})} e^{j2\pi(\frac{K-1}{2})n/K}$$

$$+ 2a_{2(\frac{K+1}{2})} e^{j2\pi(\frac{K+1}{2})n/K} + \dots + 2a_{2(K-1)} e^{j2\pi(K-1)n/K}$$

$$= \sum_{\ell=0}^{K-1} 2a_{2\ell} e^{j2\pi \ell n/K} = \sum_{\ell=0}^{\frac{N}{2}-1} 2a_{2\ell} e^{j2\pi \ell n/(N/2)}$$

h) $$y[n] = \begin{cases} x[n] & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$$

Clearly
$$y[n] = \frac{1+(-1)^n}{2} \cdot x[n]$$
as a product of two discrete time signals. Here assume that $x[n]$ is periodic of period N with N-point DTFS a_k. We might try to use the multiplication property given in table 3.2 on p. 221 of the O+W text. But a direct approach is easier.

Define $z[n] = \frac{1+(-1)^n}{2} \quad n \in \mathbb{Z}$

This is periodic of (minimal) period 2. It can, of course, be viewed as periodic with any even period.

Since $y[n] = z[n]\,x[n]$, $y[n]$ will always be periodic but in general its period will be the LCM of the period of $x[n]$ (ie N) and the period of $z[n]$ (ie. 2)

$$\text{period of } y[n] = \text{LCM}(2, N) = \begin{cases} N & \text{if } N = \text{even} \\ 2N & \text{if } N = \text{odd} \end{cases}$$

Thus must consider the cases separately.

$\boxed{\text{Case: } N = \text{even}}$

$$Y_k = \frac{1}{N} \sum_{n=0}^{N-1} \frac{1+(-1)^n}{2} x[n]\, e^{-j2\pi kn/N}$$

$$Y_k = \frac{1}{N}\sum_{n=0}^{N-1} \frac{1}{2} x[n]\, e^{-j2\pi kn/N} + \frac{1}{N}\sum_{n=0}^{N-1} \frac{1}{2} x[n] (-1)^n e^{-j2\pi kn/N}$$

Since $\frac{N}{2} = \text{integer}$

$$-1 = e^{-j2\pi (N/2)/N} = e^{-j\pi}$$

$$\Longrightarrow \quad (-1)^n = e^{-j2\pi (N/2) n/N}$$

$$Y_k = \frac{1}{N}\sum_{n=0}^{N-1} \frac{1}{2} x[n]\, e^{-j2\pi kn/N} + \frac{1}{N}\sum_{n=0}^{N-1} \frac{1}{2} x[n]\, e^{-j2\pi (k+N/2) n/N}$$

$$= \frac{1}{2} X_k + \frac{1}{2} X_{k+N/2} = \frac{1}{2} X_k + \frac{1}{2} X_{k-N/2}$$

for $k = 0, 1, 2 \ldots N-1$ (N even).

$$\boxed{\text{Case: } N = \text{odd}}$$

$$Y_k = \frac{1}{2N}\sum_{n=0}^{2N-1} \frac{1+(-1)^n}{2}\, x[n]\, e^{-j2\pi kn/2N} \qquad k = 0, 1, 2 \ldots 2N-1$$

$$= \underbrace{\frac{1}{2N}\sum_{n=0}^{2N-1} \frac{1}{2} x[n]\, e^{-j2\pi kn/2N}}_{\text{Part 1}} + \underbrace{\frac{1}{2N}\sum_{n=0}^{2N-1} \frac{1}{2} \left(e^{j2\pi N/2N}\right)^n x[n]\, e^{-j2\pi kn/2N}}_{\text{Part 2}}$$

Let's examine these parts separately.

Re : Part 1

$$\text{Part}1_k = \frac{1}{2N}\sum_{n=0}^{2N-1} \frac{1}{2}x[n]\, e^{-j2\pi kn/2N}$$

$$= \frac{1}{2N}\sum_{n=0}^{N-1} \frac{1}{2}x[n]\, e^{-j2\pi kn/2N} + \underbrace{\frac{1}{2N}\sum_{n=N}^{2N-1} \frac{1}{2}x[n]\, e^{-j2\pi kn/2N}}_{\text{c.o.v. } m=n-N}$$

$$= \frac{1}{2N}\sum_{n=0}^{N-1} \frac{1}{2}x[n]\, e^{-j2\pi kn/2N} + \frac{1}{2N}\sum_{m=0}^{N-1} \frac{1}{2}\underbrace{x[m+N]}_{=\,x[m]}\, e^{-j2\pi km/2N}\, \underbrace{e^{-j2\pi kN/2N}}_{=\,(-1)^k}$$

$$= \frac{1}{N}\sum_{n=0}^{N-1} x[n]\, e^{-j2\pi kn/2N}\underbrace{\left[\frac{1}{4} + \frac{1}{4}(-1)^k\right]}_{} = \begin{cases} 0 & \text{if } k \text{ odd} \\ \frac{1}{2} & \text{if } k \text{ even}\end{cases}$$

$\therefore$ If $k =$ odd $\Rightarrow \text{Part}1_k = 0$ and if $k =$ even, say $k = 2\ell$ then

$$\text{Part}1_k = \text{Part}1_{2\ell} = \frac{1}{2}\left(\sum_{n=0}^{N-1} x[n]\, e^{-j2\pi 2\ell n/2N}\right)$$

$$= \frac{1}{2}X_\ell$$

Re: Part 2

$$\text{Part}2_k = \frac{1}{2N}\sum_{n=0}^{2N-1} \frac{1}{2}x[n]\,\overbrace{(-1)^n}^{\left(e^{j2\pi N/2N}\right)^n}\, e^{-j2\pi kn/2N}$$

We can use the frequency shifting property of the DTFS (for period 2N), see p 221, and the result from Part 1:

$$\text{Part}2_k = \text{Part}1_{k-N}$$

We have assumed that N is odd, say $N = 2K+1$
Then

$$k - N = k - 2K - 1 = \text{an even \# when } k \text{ is odd}$$

Say $k = 2\ell + 1$

$$\Longrightarrow k - N = 2\ell + 1 - 2K - 1 = 2(\ell - K)$$

$\therefore$

$$\text{Part}2_k = 0 \quad \text{if } k \text{ is even}$$

$$\text{Part}2_{2\ell+1} = \text{Part}1_{2(\ell-K)} = \frac{1}{2}X_{\ell-K}$$

Putting it together:

$$Y_k = \begin{cases} \frac{1}{2} X_{k/2} & \text{if } k = \text{even} \\ \frac{1}{2} X_{\frac{k-N}{2}} & \text{if } k = \text{odd} \end{cases}$$

(for the case N odd).

Summary: $x[n] \longleftrightarrow X_k$ as a periodic seq. of period N

$$y[n] = \begin{cases} x[n] & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$$

<u>Case:</u> N = even. Then $y[n]$ is periodic with period N and its N-point DTFS coeffs are

$$Y_k = \frac{1}{2} X_k + \frac{1}{2} X_{k-N/2} \quad k = 0, 1 \ldots N-1$$

<u>Case:</u> N = odd. Then $y[n]$ is periodic of period 2N and its 2N-point DTFS coeffs are

$$Y_k = \begin{cases} \frac{1}{2} X_{k/2} & k \text{ even, i.e., } k = 0, 2, 4 \ldots 2N-2 \\ \frac{1}{2} X_{\frac{k-N}{2}} & k \text{ odd, i.e., } k = 1, 3, 5 \ldots 2N-1 \end{cases}$$

$\boxed{\text{O+W } 3.50}$

$x[n]$ is periodic of period $N=8$

$x[n] \longleftrightarrow X_k$

st.

1) $X_k = -X_{k-4}$

2) $x[2n+1] = (-1)^n$

$$X_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\, e^{-j2\pi kn/N}$$

$$x[n] = \sum_{k=0}^{N-1} X_k\, e^{+j2\pi kn/N}$$

First look at the implication of condition #1.

$$X_k = \frac{1}{8}\sum_{n=0}^{7} x[n]\, e^{-j2\pi kn/8}$$

$$X_{k-4} = \frac{1}{8}\sum_{n=0}^{7} x[n]\, e^{-j2\pi(k-4)n/8} = \frac{1}{8}\sum_{n=0}^{7} x[n] e^{j\pi n} e^{-j2\pi kn/8}$$

$$= \frac{1}{8}\sum_{n=0}^{7} (-1)^n x[n]\, e^{-j2\pi kn/8}$$

ie $x[n] \longleftrightarrow X_k$ and $(-1)^n x[n] \longleftrightarrow X_{k-4}$

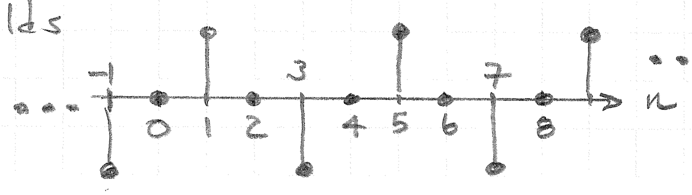
Therefore, if $X_k = -X_{k-4}$ then uniqueness of DTFS coefficients implies

$$x[n] = -(-1)^n x[n]$$

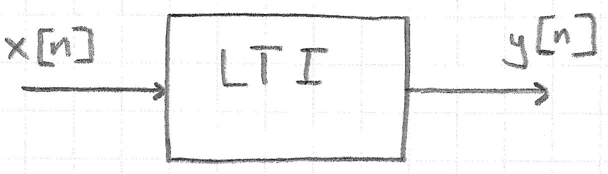
If n is even the above implies $x[n] = -x[n] \Longrightarrow$ $x[n]=0$ for n even. If n is odd the above implies $x[n] = x[n]$, which is no constraint at all. Then we turn to condition #2 which yields

$x[1] = 1 \quad x[5] = 1$

$x[3] = -1 \quad x[7] = -1 \quad \Longrightarrow$



$\boxed{\text{O + W 3.60 a,b,g}}$



For the specified input-output signal pairs can there be an LTI system as above? If such exists, is it unique? If unique determine the frequency response.

(a) $x[n] = \left(\frac{1}{2}\right)^n$ and $y[n] = \left(\frac{1}{4}\right)^n$

If the system were LTI then $y[n]$ would have to be of the form:

$$y[n] = \underbrace{H\left(\tfrac{1}{2}\right)}_{\text{a constant, indep. of } n.} \cdot \left(\tfrac{1}{2}\right)^n \qquad \forall n \in \mathbb{Z}$$

Note $y[n] = \left(\frac{1}{4}\right)^n$ does not satisfy. Therefore, there is no LTI system that has $x[n]$ as input and $y[n]$ as output.

(b) $x[n] = \left(\frac{1}{2}\right)^n u[n]$ and $y[n] = \left(\frac{1}{4}\right)^n u[n]$.

There is an easy way to do this with system functions and the discrete-time Fourier Transform, but we have not yet covered these. So lets look directly for an impulse response (without loss of generality, a causal $h[n]$) st

$$y[n] = h * x[n]$$

$$y[n] = \sum_{k=0}^{n} h[k]\, x[n-k] = \sum_{l=0}^{n} x[l]\, h[n-l]$$

Then substituting:

$$\left(\frac{1}{4}\right)^n = \sum_{k=0}^{n} h[k] \left(\frac{1}{2}\right)^{n-k} = \left(\frac{1}{2}\right)^n \sum_{k=0}^{n} h[k] \left(\frac{1}{2}\right)^{-k}$$

for $n = 0, 1, 2 \ldots$ Simplify a little to get

$$\left(\frac{1}{2}\right)^n = \sum_{k=0}^{n} h[k]\, 2^k \qquad n = 0, 1, 2 \ldots$$

$$\begin{array}{lrcl}
n=0 & 1 & = & h[0] \\
n=1 & \frac{1}{2} & = & h[0] + 2h[1] \\
n=2 & \frac{1}{4} & = & h[0] + 2h[1] + 4h[2] \\
n=3 & \frac{1}{8} & = & h[0] + 2h[1] + 4h[2] + 8h[3] \\
\vdots & & &
\end{array}$$

It is easy to see that the above system of eqns has a unique solution (its a triangular system). To find the solution we could just guess or note

$$\left(\frac{1}{2}\right)^n = \underbrace{h[0] + 2h[1] + \cdots + 2^{n-1} h[n-1]}_{\left(\frac{1}{2}\right)^{n-1}} + 2^n h[n]$$

$$\Rightarrow \left(\tfrac{1}{2}\right)^n = \left(\tfrac{1}{2}\right)^{n-1} + 2^n h[n]$$

$$\Rightarrow h[n] = \frac{\left(\tfrac{1}{2}\right)^n - \left(\tfrac{1}{2}\right)^{n-1}}{2^n} \qquad n \geq 1 \quad (\text{\& } h[0] = 1)$$

$$= \frac{\left(\tfrac{1}{2}\right)^n [1-2]}{2^n}$$

$$= \begin{cases} -\left(\tfrac{1}{4}\right)^n & n \geq 1 \\ 1 & n = 0 \end{cases}$$

Is unique. Can solve for system function using

$$H(e^{j\omega}) = \sum_{k=0}^{\infty} h[k] e^{-j\omega k}$$

$$= 1 - \sum_{k=1}^{\infty} \left(\tfrac{1}{4}\right)^k e^{-j\omega k}$$

$$= 1 - \left[\frac{1}{1 - \frac{1}{4}e^{-j\omega}} - 1\right]$$

$$H(e^{j\omega}) = 2 - \frac{1}{1-\frac{1}{4}e^{-j\omega}}$$

$$= \frac{1-\frac{1}{2}e^{-j\omega}}{1-\frac{1}{4}e^{-j\omega}}$$

(g) $x[n] = \cos(\pi n/3)$; $y[n] = \cos(\pi n/3) + \sqrt{3}\sin(\pi n/3)$

$$= \frac{e^{-j\pi n/3} + e^{+j\pi n/3}}{2} \qquad = \frac{e^{-j\pi n/3} + e^{+j\pi n/3}}{2}$$

$$+ \frac{\sqrt{3}}{j2}\left(e^{+j\pi n/3} - e^{-j\pi n/3}\right)$$

$\therefore$ Periodic of period $N=6$

$X_1 = X_{-1} = \frac{1}{2}$ (the other coeffs are zero)

$$Y_{-1} = \frac{1}{2} - \frac{\sqrt{3}}{j2} = \frac{1}{2} + j\frac{\sqrt{3}}{2}$$

$$Y_1 = \frac{1}{2} + \frac{\sqrt{3}}{j2} = \frac{1}{2} - j\frac{\sqrt{3}}{2}$$

From our prev. work know there is such an LTI system and its transfer function must satisfy

$$\frac{1}{2} + j\frac{\sqrt{3}}{2} = H(e^{-j2\pi/6})\,\frac{1}{2}$$

and

$$\frac{1}{2} - j\frac{\sqrt{3}}{2} = H(e^{+j2\pi/6})\,\frac{1}{2}$$

ie $H(e^{j2\pi/6}) = 1 - j\sqrt{3}$

This only specifies one frequency so the LTI system is <u>not</u> <u>unique</u>.