(continue) $y[n] = a\,y[n-1] + x[n]$

We got $\rightarrow h[n] = a^n u[n]$

- For $a=1$ $\quad h[n] = u[n]$ (1)

  $y[n] = y[n-1] + x[n]$

  This is the same system as

  $y[n] = \sum_{k=-\infty}^{\infty} x[k]$

"Two different realization of the same system"

- (side-note) the Inverse system has the difference eq.

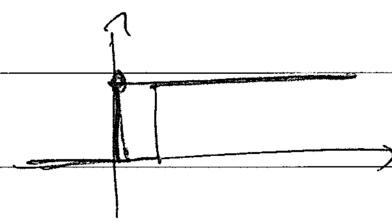
  $y[n] = x[n] - x[n-1]$

  $\rightarrow h[n] = \delta[n] - \delta[n-1]$ (2) : impulse response

$u[n]^{(1)} * (\delta[n] - \delta[n-1])^{(2)} = u[n] - u[n-1] = \delta[n]$

Inverse

" $h[n] * h_I[n] = \delta[n]$ "



# Chap 3. Fourier Series for Periodic Signals.

## 1) CT Fourier Series

- consider $x(t) = x(t+T)$ for all $t$.
- Fourier series expansion for periodic signal with period $T$

$$\boxed{x(t) = \sum_{k=-\infty}^{\infty} a_k S_k(t)}$$

, $a_k$ : Fourier series coefficients
  ( complex amplitudes of the sinewaves)

, $S_k(t) = e^{j 2\pi \frac{k}{T} t}$
  ( orthogonal sinewaves forming a complete basis for all periodic signals with period $T$ )

$\frac{1}{T}$ = fundamental freq. (Hz)

$\frac{k}{T}$ = k-th harmonic freq. ($k$ : integer)

$S_k(t)$ is periodic with period $\frac{T}{k}$ and is thus periodic with period $T$.

complex conjugate "$(a+jb)^* = a-jb$"

- $s_k(t)$ and $s_l(t)$ are orthogonal if

$$\int_{-T/2}^{T/2} s_k(t)\, s_l^*(t)\, dt = 0 \quad \text{for } k \neq l$$

(proof):
$$\int_{-T/2}^{T/2} e^{j2\pi \frac{k}{T}t} \cdot e^{-j2\pi\frac{l}{T}t}\, dt$$

$$= \int_{-T/2}^{T/2} e^{j2\pi\frac{(k-l)}{T}t} = \frac{1}{j2\pi\frac{(k-l)}{T}}\, e^{j2\pi\frac{(k-l)}{T}t}\Big|_{-T/2}^{T/2}$$

$$= \frac{T}{j2\pi(k-l)}\left\{ e^{j\cancel{2}\pi\frac{(k-l)}{\cancel{T}}\cdot\frac{\cancel{T}}{\cancel{2}}} - e^{-j\cancel{2}\pi\frac{(k-l)}{\cancel{T}}\cdot\frac{\cancel{T}}{\cancel{2}}}\right\}$$

$$= \frac{T}{j2\pi(k-l)}\left\{ (\underbrace{e^{j\pi}}_{=-1})^{k-l} - (\underbrace{e^{-j\pi}}_{=-1})^{k-l}\right\} = 0, \quad k\neq l$$

- More generally: $\int_{t_0}^{t_0+T} s_k(t)\, s_l^*(t)\, dt = \begin{cases} T & \text{if } k=l \\ 0 & \text{if } k\neq l \end{cases} = T\,\delta[k-l]$

for any constant $t_0$

- As a result, the FS* coeff. (complex amplitudes) may be found as: (since $x(t) = \sum_{k=-\infty}^{\infty} a_k s_k(t)$ : FS)

$$a_k = \frac{1}{T}\int_{t_0}^{t_0+T} x(t)\, s_k^*(t)\, dt$$

$$= \frac{1}{T}\int_{t_0}^{t_0+T} \left(\sum_{l=-\infty}^{\infty} a_l s_l(t)\right) s_k^*(t)\, dt$$

$$= \sum_{l=-\infty}^{\infty} a_l \cdot \frac{1}{T}\int_{t_0}^{t_0+T} s_l(t)\, s_k^*(t)\, dt$$

$$= \sum_{l=-\infty}^{\infty} a_l \cancel{\frac{1}{T}} \cdot \cancel{T}\, \underbrace{\delta[k-l]}_{l=k} \qquad = \cdots a_0 \underbrace{\delta[k-0]}_{=0} + a_1 \underbrace{\delta[k-1]}_{=0} \cdots + a_k \underbrace{\delta[k-k]}_{=1} + \cdots \underbrace{\cdots}_{0}$$

$$= a_k \qquad (= a_k)$$

- In particular $a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\cdot e^{-j2\pi \frac{k}{T} t}\, dt$

* FS : Fourier Series.
* coeff : coefficient

## 2) DT Fourier Series

- Fourier Series Representation of DT periodic signals.

$$x[n] = x[n+N] \quad \text{for all } n., \quad N: \text{Integer}$$

- Recall features of DT sinewaves

only periodic if $\frac{w_0}{2\pi} = \frac{m}{N}$ (rational), $m$ and $N$ : integers.

$e^{j(w_0 + l 2\pi)} = e^{jw_0 n}$, $l$ : Integer

( DT freq. are only unique over 2π interval, e.g. $[0, 2\pi)$ or $[-\pi, \pi)$ )

- As a result, for $x[n] = x[n+N]$ only need to sum $N$ DT sinewaves at $N$ freq. equi-spaced over some 2π interval.

→ $[0, 2\pi)$

Def. $$\boxed{x[n] = \sum_{k=0}^{N-1} a_k \underbrace{s_k[n]}_{e^{j2\pi \frac{k}{N} n}}}$$

The $N$ DT sinewaves $s_k[n]$ are orthogonal :

$$\sum_{n=0}^{N-1} s_k[n]\, s_l^*[n] = N\,\delta[k-l] = \begin{cases} N, & \text{if } k=l \\ 0, & \text{if } k \neq l \end{cases}$$

$k, l \in [0, N-1)$ : integers.

(proof) $\sum_{n=0}^{N-1} e^{j2\pi\frac{k}{N}n} e^{-j2\pi\frac{l}{N}n}$

$$= \sum_{n=0}^{N-1} \left(e^{j2\pi\frac{(k-l)}{N}}\right)^n$$

$$= \frac{1 - e^{j2\pi\frac{(k-l)}{\cancel{N}}\cancel{N}}}{1 - e^{j2\pi\frac{(k-l)}{N}}} = \frac{1 - (\underbrace{e^{j2\pi}}_{=1})^{k-l}}{1 - e^{j2\pi\frac{k-l}{N}}}$$

$$= \begin{cases} 0 & \text{if } k \neq l \\ N & \text{if } k = l \end{cases} = N\delta[k-l]$$

- As a result of the orthogonality, the FS coeff (complex amplitudes of the sinewaves) may be found as:

$$a_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\, s_k^*[n]$$

$$= \frac{1}{N}\sum_{n=0}^{N-1} x[n]\, e^{-j2\pi\frac{k}{N}n}, \quad k = 0, 1, \cdots, N-1$$

$$= \frac{1}{N}\sum_{n=n_0}^{n_0+N-1} x[n]\, e^{-j2\pi\frac{k}{N}n} \quad \text{(can take any period)}$$