1. Can you find a sequence $\{a_n\}$ of real numbers where there are infinitely many n with $a_n > \limsup a_n$?

2. (a) Given a sequence $\{x_1, x_2, ...\}$ of real numbers, can you find a sequence, say, $\{a_n\}$ with $E = \{\text{limit points of } \{a_n\}\} = \{x_1, x_2, ...\}$? Hint: First consider E not closed, and then consider E closed.

   (b) Can you find a sequence $\{a_n\}$ of real numbers where $E = \{\text{limit points of } \{a_n\}\} = \mathbb{R}$?

3. Let $a_n, b_n \in \mathbb{R}$. Is it true that $\limsup a_n b_n = \limsup a_n \limsup b_n$? What if a_n and b_n are non-negative? Can you prove or disprove any inequality?

4. Given a sequence $\{an\}$, let $\mathcal{A} = \{\{a_{n_k}\}; \{a_{n_k}\}\text{a subsequence of } \{an\}\}$. What are the possible cardinalities of $\mathcal{A}$?

5. Rudin's definition of the limit supremum is

$$\limsup a_n = \sup E$$

   with E the set of limit points of $\{a_n\}$. Show this is equivalent to the following definition

$$\limsup a_n = \lim_{n\to\infty} \sup_{k\geq n} a_k.$$

   Hint: $\sup_{k\geq n} a_k$ decreasing in n. Remark: This second definition is often easier in practice.

6. Given a sequence or real numbers $\{a_n\}$, let $A = \{\{a_{n_k}\}\}$ i.e., the family of subsequences. Is A countable or uncountable?

7. Suppose $a_n > 0, a_n \to a > 0, 0 < \lambda < \dfrac{1}{a}$ and show

$$\sum_0^\infty \lambda^n (a_0 a_1 \dots a_n)$$

   converges absolutely. (hint : compare to a geometric series).

8. In a metric space, show $\{a_n\}$ converges iff every subsequence $\{a_{n_k}\}$ has a further convergent subsequence $\{a_{n_{k_j}}\}$.

9. Suppose X is a complete metric space, and $\{G_n\}$ is a family of open sets, and each G_n is dense, i.e. $\bar{G_n} = X$. Show $\cap G_n$ is nonempty.