Countability

1. Countable or uncountable (with proof)?

   (a) $\oplus_{\mathbb{N}}\mathbb{Q} = \{(q_1, q_2 ...) \in \mathbb{Q} \times \mathbb{Q} \times \mathbb{Q} \times ... :$ only finitely many q_i are non-zero.$\}$.

   (b) $\Pi_{\mathbb{N}}\mathbb{Q} = \{(q_1, q_2 ...) \in \mathbb{Q} \times \mathbb{Q} \times \mathbb{Q} \times ...\}$

Sets and mappings

2. Let $f : A \to B$, A and B subsets or $\mathbb{R}^n$. True or false with proof:

   (a) $\cup f^{-1}(B_j) = f^{-1}(\cup B_j)$

   (b) $\cap f^{-1}(B_j) = f^{-1}(\cap B_j)$
   (Here the $B_j \subseteq B$ and if the arbitrary union seems troubling, use only B_1 and B_2.)

Convexity

3. Let A be a convex set in $\mathbb{R}^n$ and show A^o and $\bar{A}$ are convex also.

4. (a) Set $||x||_p := (\sum_{j=1}^n |x_j|^p)^{1/p}$ for every $x = (x_1, ..., x_n) \in \mathbb{R}^n$ and every $1 \leq p < \infty$. Show $d_p(x, y) := ||x - y||_p$ is a metric. (Notice, $p = 2$ is our usual Euclidean metric on $\mathbb{R}^n$.

   (b) Now on $\mathbb{R}^2$ let $A := \{x : ||x||_1 \leq 1\}$, and show A is convex in $(\mathbb{R}^2, d_1)$ but not in the usual metric, $(\mathbb{R}^n, d_2)$.

5. (Lempert) Let A be an open, non-empty, convex set in $\mathbb{R}^n$, and show $A_r := \{x \in A : d(x, A^c) < r\}$ is also convex.

6. For any open $G \subseteq \mathbb{R}$, show G is a countable union of balls, i.e. $\exists\{B_{r_j}(x_j)\}_{j\in\mathbb{N}}$ with $G = \cup_j B_{r_j}(x_j)$.

Metric Spaces and Topology

7. True of False:

   If (X, d) is a non-empty metric space, and $x \in X$ then $\overline{N_1(x)} = \{y : d(x, y) \leq 1\}$

8. Recall, if G_α are open, and C_α are closed, then we know $\cup G_\alpha$ is open and $\cap C_\alpha$ is closed.

   Exhibit an example of open sets, G_α in say $\mathbb{R}^n$ such that $\cap G_\alpha$ is not open.

   Notice, by de Morgan's Law, we obtain a sequence of closed C_α with $\cup C_\alpha$ not closed.

9. Consider $\mathbb{Z}$ the integers as a subspace of $\mathbb{R}$. Let $A \subseteq \mathbb{Z}$. Is A open in $\mathbb{Z}$ (i.e. is A open relative to $\mathbb{Z}$?) Is A open in $\mathbb{R}$. What about closed relative to $\mathbb{Z}$? $\mathbb{R}$?

10. Prove a nonempty perfect subset of $\mathbb{R}$ is uncountable.

11. Does there exist a dense proper open subset of $\mathbb{R}$?

12. For each $j \in \mathbb{N}$ let G_j be a dense open subset of $\mathbb{R}^n$. Show $\cap_{j=1}^{\infty} G_j$ is non-empty.