Real Analysis Qual Prep Summer 2009

Assignment 4: Integration

1. True or false: If f is a non-negative function defined on $\mathbf{R}$ and

$$\int_{\mathbf{R}} f\,dx < \infty,$$

then $\lim_{|x|\to\infty} f(x) = 0$.

2. Let $(X, \mathcal{F}, \mu)$ be a measure space and $f \in L(\mu)$. Show that $\{f \neq 0\}$ is σ-finite.

3. True or false: Let $(X, \mathcal{F}, \mu)$ be a finite measure space, $\{f_n\}$ a non-increasing sequence of non-negative functions which converges to f.
Then $\int_X f_n\,d\mu \to \int_X f d\mu$.

4. Let f be a non-negative measurable function on $\mathbf{R}$. Prove that if

$$\sum_{n=-\infty}^{\infty} f(x+n)$$

is integrable, then $f = 0$ a.e.

5. Let $\{r_1, r_2, ...\}$ be an enumeration of $\mathbf{Q} \cap [0,1]$, and let $f(x) = \sum_{\{n: x > r_n\}} 2^{-n}$.

Compute $||f||_1$.

6. Prove that the sum

$$\sum_{n=0}^{\infty} \int_0^{\pi/2} (1 - \sqrt{\sin x})^n \cos x\,dx$$

converges to a finite limit, and find its value.

7. Let f be a continuous function on $I = [-1, 1]$ with the property that $\int_I x^n f(x)\,dx = 0$ for $n = 0, 1, 2, ...$ Show that f is identically zero.

8. If $f \in L_1[0,1]$, show that given $\epsilon > 0$ there exists $\delta > 0$ such that $\mu(A) < \delta$ implies $\int_A |f| < \epsilon$.

9. (a) Let $f \in L^1(0,\infty)$. Prove that there exists a sequence $x_k \nearrow \infty$ such that

$$\lim_{k\to\infty} |f(x_k)| = 0.$$

(b) Let $f \in L^1(\mathbf{R}^n)$ with $n \geq 2$. Prove that there exists a sequence $R_k \nearrow \infty$ such that

$$\lim_{k\to\infty} R_k \int_{S(0,R_k)} |f|\, d\sigma = 0,$$

where $S(0,r) = \{x \in \mathbf{R}^n \,|\, |x| = r\}$, and $d\sigma$ represents the (n-1)-dimensional Lebesgue measure induced on the sphere $S(0,r)$.

10. Prove for every $\xi \in \mathbf{R}^n$ prove the existence of the limit

$$\lim_{k\to\infty} \int_{\mathbf{R}^n} \frac{e^{-2\pi i<\xi,x>} e^{-|x|^2}}{(k^{-n} + k|x|^2)^{(1/2)(n+1)}}\, dx,$$

and compute it explicitly in terms of the $(n-1)$-dimensional measure σ_{n-1} of the unit sphere in $\mathbf{R^n}$, and the Beta function

$$B(x,y) := 2\int_0^{\pi/2} \cos(\theta)^{2x-1} \sin(\theta)^{2y-1}\, d\theta,\ x,y > 0.$$

Remark: Note carefully that you are not required to know or write explicity the value of σ_{n-1}.

11. Let $f_1, f_2, ...$ be functions on $\mathbf{R}^n$ such that

$$\int_{\mathbf{R}^n} f_k = 1,\ k \geq 1, \text{ and } 0 \leq f_k \leq \tfrac{1}{k}.$$

Prove $\int_{\mathbf{R}^n} \sup\limits_{k\geq 1} f_k = \infty$.

12. Let f be a function on $(-\infty,\infty)$ such that given $\epsilon > 0$ there is a polynomial $p(x)$ such that $|p(x) - f(x)| < \epsilon$ for all $x \in \mathbf{R}$. Show that f is a polynomial.

13. Let f be a real-valued measurable function on $[a,b]$ such that $\int_a^b f^n\, dx = c$ for $n = 2,3,4$. Show that $f = \chi_A$ a.e. for some measurable set $A \subset [a,b]$.

14. Let $(X,\mathcal{F},\mu)$ be a finite measure space, and f a measurable extended real-valued function defined on X. Show that $f \in L(\mu)$ if and only if

$$\sum_{k=1}^{\infty} \mu\{|f| \geq k\} < \infty.$$

15. Let f be a continuous function on $[-1, 1]$. Find

$$\lim_n \int_{-1}^{1} f(x)(1 - n|x|)\, dx.$$

16. True or false: If $f \in L^1(\mathbf{R})$ and $||f\chi_A||_1 = 0$ for all measurable sets A satisfying $\mu(A) = \pi$, then $f = 0$ a.e.

17. Show that if $f_n \to f$ a.e., $f_n \geq 0$, $g_n \to g$ a.e., $f_n \leq g_n$ a.e., and $||g_n||_1 \to ||g||_1$, then $||f_n||_1 \to ||f||_1$.

18. Calculate $\lim_{t\to 0^+} \int_0^1 \frac{e^{-t\ln x}-1}{t}\, dx$. (Hint: don't think too hard).

19. Let $\alpha \geq 1$ and compute, with justification,

$$\lim_{n\to\infty} \int_0^{\pi} n \ln\left[1 + \left(\frac{\sin x}{n}\right)^{\alpha}\right] dx.$$

20. Let $(X, \mathcal{F}, \mu)$ be a measure space and let $f : X \to [0, \infty]$ be measurable with $f \in L^1(X, \mu)$. Compute

$$\lim_{n\to\infty} \int_X n \arctan[(f/n)^{\alpha}]\, d\mu,\ \alpha \in (0, \infty).$$

21. Determine whether the limits exist, and if so, compute their values:

    (a) $\displaystyle\lim_{n\to\infty} n \int_{-1}^{1} e^{-(\frac{nx+1}{n})^2} - e^{-x^2}\, dx$

    (b) $\displaystyle\lim_{n\to\infty} n \int_{-\infty}^{\infty} e^{-(\frac{nx+1}{n})^2} - e^{-x^2}\, dx$

22. Define $F(t) = \int_0^{\infty} t^3 e^{-t^2 x}\, dx$. Show that $F(t)$ is differentiable, and compute $F'(0)$.