ECE 495N EXAM II

Friday, Nov.6, 2009

NAME : SOLUTION

PUID # : ______

CLOSED BOOK

Useful relations

$$h(\vec{k}) = \sum_m [H_{nm}] \exp(i\vec{k}.(\vec{d}_m - \vec{d}_n))$$ ***Bandstructure***

$$D(E) = \sum_k \delta(E - \varepsilon(\vec{k}))$$ ***Density of states***

$$M(E) = \sum_k \delta(E - \varepsilon(\vec{k}))\, \pi \hbar v_z(\vec{k})/L$$ ***Density of modes***

$$C_Q = q^2 \int_{-\infty}^{+\infty} dE \left(-\frac{\partial f}{\partial E}\right) D(E)$$ ***Quantum Capacitance***

$$G_B = \frac{q^2}{h} \int_{-\infty}^{+\infty} dE \left(-\frac{\partial f}{\partial E}\right) M(E)$$ ***Ballistic Conductance***

$$f(E) = \frac{1}{e^{(E-\mu)/kT} + 1}$$ ***Fermi function***

$$\hbar v_z = \frac{\partial \varepsilon}{\partial k_z}$$ **Group velocity**

Please show all work and write your answers clearly.
This exam should have six pages.

Problem 1	**[p. 2]**	**8 points**
Problem 2	**[p. 3,4]**	**9 points**
Problem 3	**[p. 5,6]**	**8 points**
Total		**25 points**

Problem 1: We have seen in class that for a periodic solid with lattice spacing 'a' and

$H_{n,n} = \varepsilon$, $H_{n,n+1} = t$ and $H_{n,n-1} = t$

(all other $H_{n,m}$ being zero) the energy eigenvalues are given by $E = \varepsilon + 2t \cos ka$.

How will this E(k) relation be modified

if $H_{n,n} = \varepsilon$, $H_{n,n+1} = t$, $H_{n,n-1} = t$, $H_{n,n+2} = t'$, $H_{n,n-2} = t'$ (all other $H_{n,m}$ being zero) ?

$$E = \varepsilon + t\left(e^{ika} + e^{-ika}\right) + t'\left(e^{i2ka} + e^{-i2ka}\right)$$

$$= \varepsilon + 2t\cos ka + 2t'\cos 2ka$$

$$\sum_m H_{nm}\, e^{ik(m-n)a}$$

Problem 2: Consider a large two-dimensional conductor with a parabolic dispersion relation

$$\varepsilon(\vec{k}) = E_c + \frac{\hbar^2(k_x^{\,2} + k_y^{\,2})}{2m}$$

with an equilibrium electrochemical potential $\mu - E_c >> k_B T$.

a) Find the density of states D(E). Your answer should be in terms of E_c, m, length L, width W and fundamental constants.

b) If this conductor is used as one plate of a parallel plate capacitor with insulator thickness "t", find "t" such that the quantum capacitance equals the electrostatic capacitance $\varepsilon_0 \varepsilon_r LW / t$, $\varepsilon_0 \varepsilon_r$ being the permittivity of the insulator.

a) $$D(E) = \sum_{\vec{k}} \delta(E - \varepsilon(\vec{k}))$$

$$= \int_0^{2\pi} d\theta \int_0^{\infty} dk\, k \cdot \frac{LW}{4\pi^2}\, \delta(E - \varepsilon)$$

$$= \frac{LW}{2\pi} \int_{E_c}^{\infty} \frac{d\varepsilon\, m}{\hbar^2}\, \delta(E - \varepsilon)$$

$$= \frac{m LW}{2\pi\hbar^2}\, \theta(E - E_c)$$

b) $$C_Q = q^2 \int dE \left(-\frac{\partial f}{\partial E}\right) D(E)$$

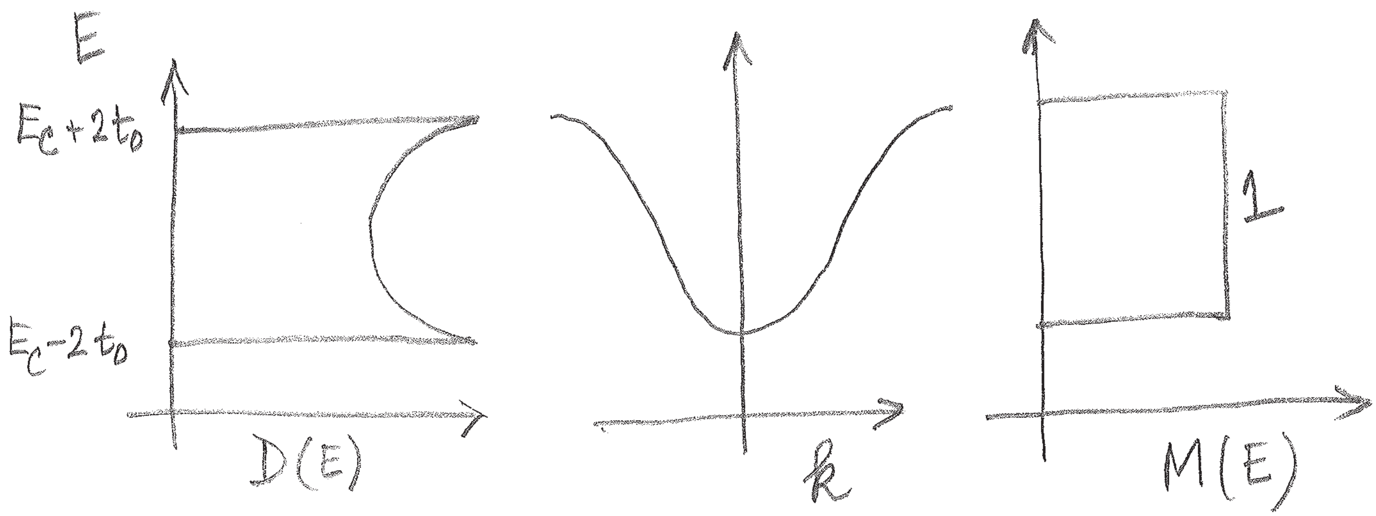
$$= \frac{m q^2}{2\pi\hbar^2} \cdot \cancel{LW} = \frac{\varepsilon_r \varepsilon_0 \cancel{LW}}{t}$$

$$t = \frac{\varepsilon_r \varepsilon_0 \cdot 2\pi\hbar^2}{m q^2}$$

Problem 3: Consider a long one-dimensional conductor with a cosine dispersion relation ($t_0 > 0$)

$$\varepsilon(k) = E_c - 2t_0 \cos(ka)$$

Find and sketch a) the density of modes M(E) and b) the density of states D(E). Your answer should be in terms of the length L, a, t_0, E, E_c and fundamental constants.



$$D(E) = 2\int_0^{\pi/a} \frac{dk \cdot L}{2\pi}\, \delta(E - \varepsilon)$$

$$= \frac{L}{\pi} \int_{E_c - 2t_0}^{E_c + 2t_0} d\varepsilon\, \frac{dk}{d\varepsilon}\, \delta(E - \varepsilon) = \frac{L}{\pi \hbar v(E)}$$

where $\hbar v = 2 a t_0 \sin ka$

$$= 2 a t_0 \sqrt{1 - \cos^2 ka}$$

$$\uparrow$$

$$\left(\frac{E - E_c}{2t_0}\right)^2$$

$$= a\sqrt{(2t_0)^2 - (E-E_c)^2}$$

$$D(E) = \frac{L}{\pi \hbar a} \frac{1}{\sqrt{(2t_0)^2 - (E-E_c)^2}}$$

$$M(E) = 2\int_0^{\pi/a} \frac{dk \cdot L}{2\pi}\, \delta(E-\varepsilon) \cdot \frac{\pi \overbrace{\hbar v}^{\frac{d\varepsilon}{dk}}}{L}$$

$$= \int_{E_c - 2t_0}^{E_c + 2t_0} d\varepsilon\, \delta(E-\varepsilon)$$

$$= \Theta\left(E - \overline{E_c + 2t_0}\right)$$

$$- \Theta\left(E - \overline{E_c - 2t_0}\right)$$