1. Representation of a CT signal using samples

What is sampling?

process of measuring the value of $x(t)$ at discrete values of times

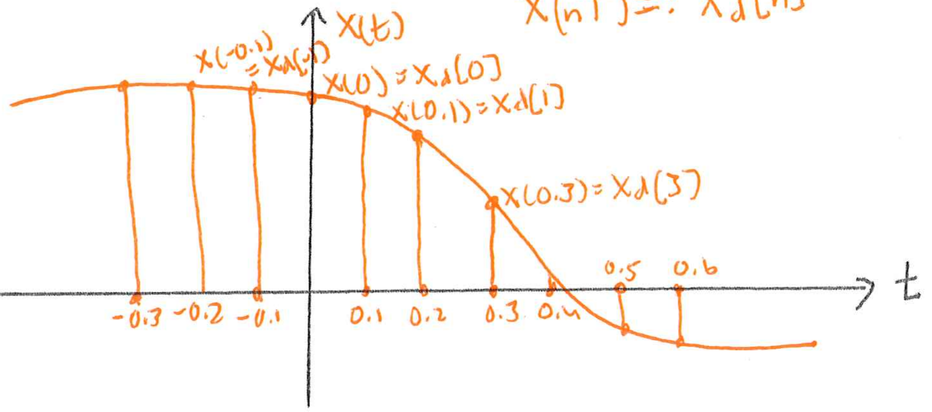
$$\ldots, t_{-3}, t_{-2}, t_{-1}, t_0, t_1, t_2, t_3 \ldots$$

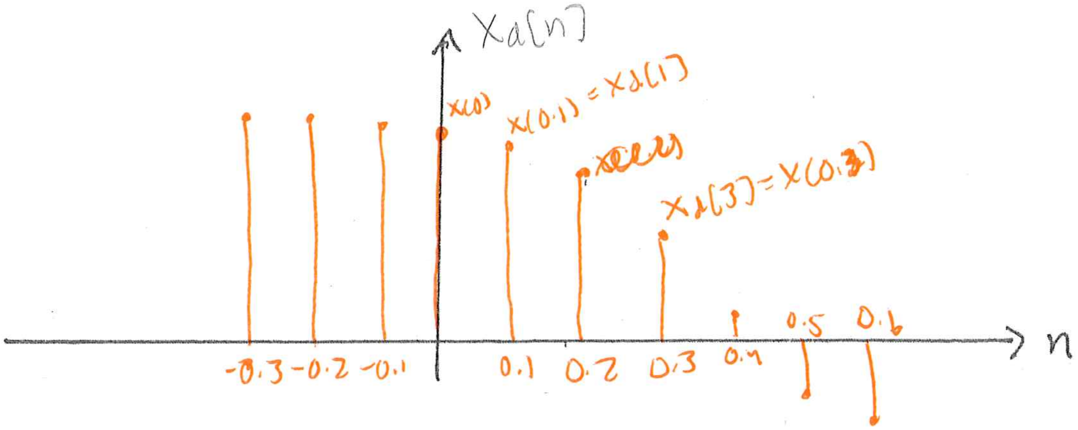
$$\ldots -14.3, -11.8, -8.3, -2.3, 5.4 \ldots$$

$t_n = nT$

$X(nT) =: X_d[n]$

illustration:





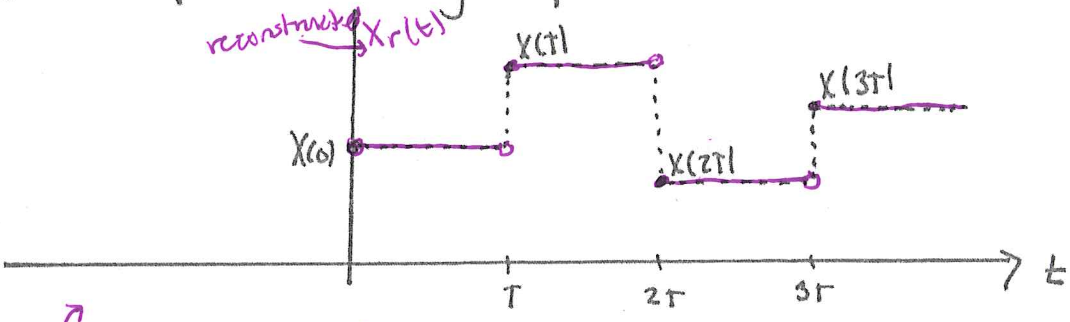
The problem: Given $y[n] = x(nT)$, $n \in \mathbb{Z}$.

Can one reconstruct $x(t)$?

In general, it is impossible to reconstruct $x(t)$ from a sampling $x_d[n] = x(nT)$ regardless of how small T is.

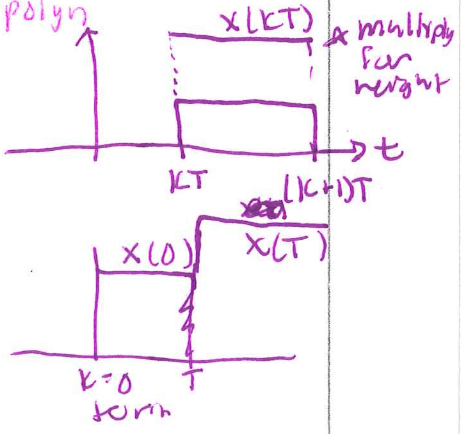
T = sampling period

Example: Interpolation using step functions.



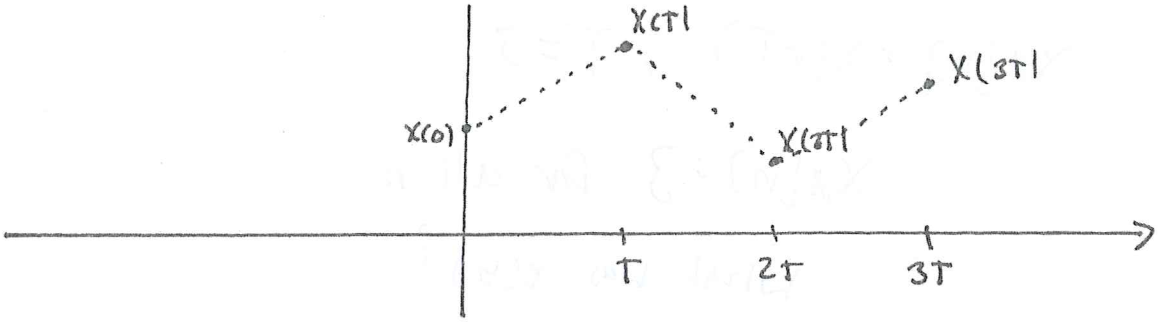
piecewise constant reconstruction deg 0 polyn

$$x_r(t) = \sum_{k=-\infty}^{\infty} \underset{\overset{\uparrow}{\text{sample value}}}{x(kT)} \underbrace{\left(u(t-kT) - u(t-(k+1)T)\right)}_{g_k(t) = g(t-kT)}$$



$x(0)(u(t) - u(t-T))$

Example: Interpolation using straight lines



piece-wise linear reconstruction. (deg 1 polynomial)

∴ There are infinitely many ways to interpolate the samples and get an approximation for $x(t)$. All these approximations could have been the original $x(t)$. So it is impossible to reconstruct $x(t)$ from samples, unless restrictions are put on $x(t)$.

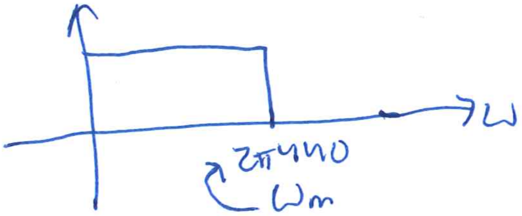
— piece-wise quadratic reconstruction (deg 2 polyn) x(t) unknown

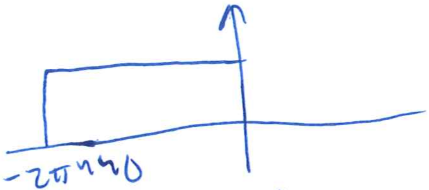
$x_d[n] = x(nT)$, T = 3

$x_d[n] = 3$ for all n

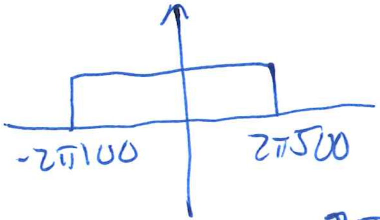
what was x(t)?

x(t) = 3 would be the "simplest" possible x(t) fitting the observation

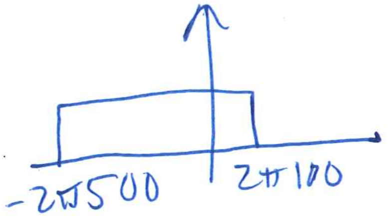




$\omega_m = 2\pi 440$



$\omega_m = 2\pi 500$



$\omega_m = 2\pi 500$

$\omega_s > 2\omega_m = 2\pi 1000$

* sample ABOVE!

2. CT signal reconstruction from samples

Sampling Theorem:

$\omega > \omega_m$
$\omega < -\omega_m$

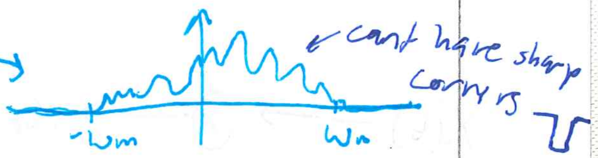
Let $\omega_M > 0$.

Let $x(t)$ be a signal with $X(\omega)=0$ when $|\omega| > \omega_M$.

Let $T>0$ and consider the sampling $x_d[n] = x(nT)$ for $n = 0, \pm1, \pm2, \pm3, \ldots$.

If $T < \frac{1}{2}\left(\frac{2\pi}{\omega_M}\right)$, then $x(t)$ can be reconstructed from $x_d[n]$.

If a signal is "band-limited" and if T is small enough (sampling freq is high enough) then one can recover $x(t)$ from the sampling $x_d[n] = x(nT)$



← can't have sharp corners

T = sampling period

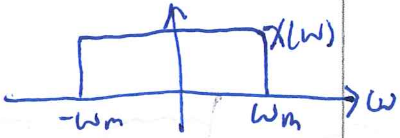
$\frac{2\pi}{T} = \omega_s$ sampling frequency

$T < \frac{1}{2}\frac{2\pi}{\omega_m} \Leftrightarrow \frac{2\pi}{T} > 2\omega_m$

$2\omega_m < \frac{2\pi}{T} \Leftrightarrow \omega_s > 2\omega_m$

ω_s i.e. sample above the Nyquist rate

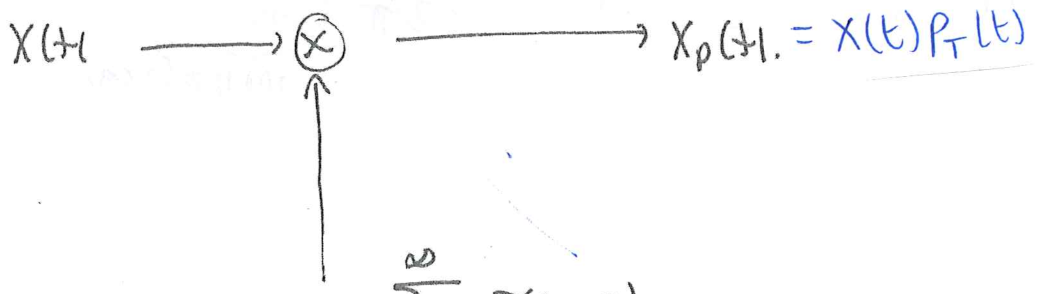
[$2\omega_m$ is the "Nyquist rate"] of $x(t)$



Why is this theorem true?

Another way to store values of samples $x_d[n] = x(nT)$

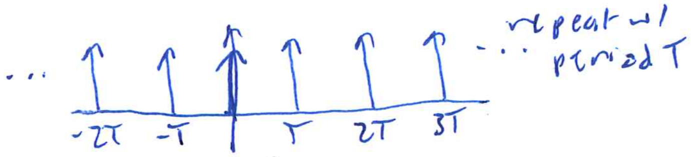
"impulse-train sampling"



$$x_p(t) = x(t)p_T(t)$$

$$p_T(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT)$$

comb



... repeat w/ period T

$W_s = \frac{2\pi}{T}$

$\frac{W_s}{2} = \frac{\pi}{T}$

$W_s > 2W_m$

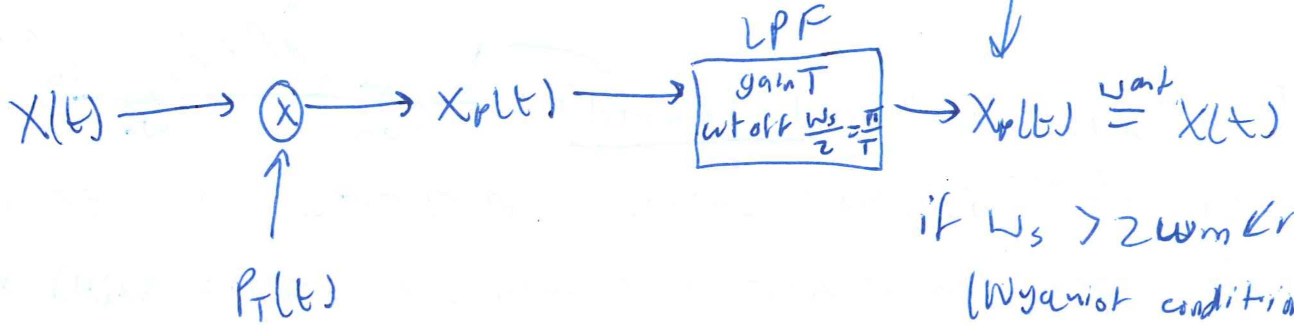
$\frac{W_s}{2} > W_m$

$W_m < \frac{W_s}{2}$

$W_m < \frac{W_s}{2} = \frac{\pi}{T}$

always band lim

$W_m \le \frac{W_s}{2}$

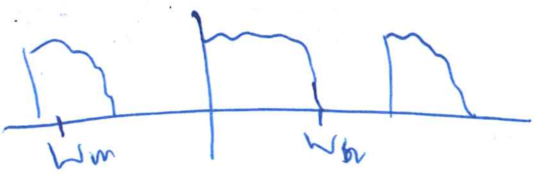


if $W_s > 2W_m$ max

(Nyquist condition)

then

$X_p(t) = X(t)$



Ws your choice

Ws = sampling freq

$$\boxed{W_s = \frac{2\pi}{T}}$$

$T = 10^{-3}$

$W_s = 2\pi \cdot 10^3$

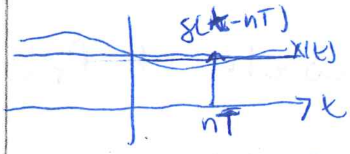
$= 2\pi \cdot \underline{1000}$

1000 Hz freq

what does $x_p(t)$ look like?

$$x_p(t) = x(t) p_T(t)$$

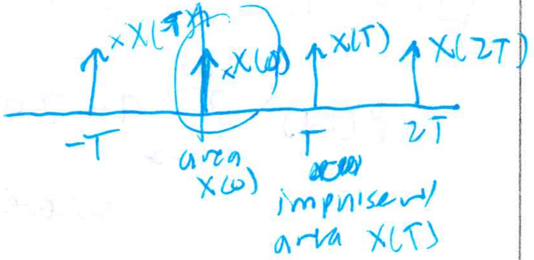
$$= x(t) \sum_{n=-\infty}^{\infty} \delta(t-nT)$$



$$= \sum_{n=-\infty}^{\infty} x(t)\delta(t-nT)$$

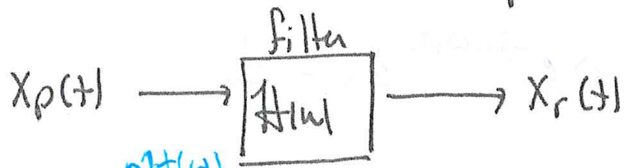
$$= \sum_{n=-\infty}^{\infty} x(nT)\delta(t-nT)$$

$x(nT) =: x_d[n]$

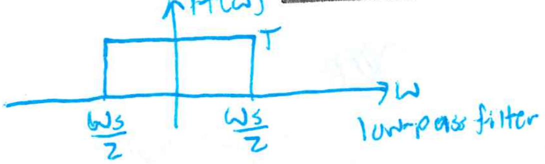


helpful w/ hw

We can recover x(t) from $x_p(t)$ as follows:



$= x(t)$ if Nyquist cond. is satisfied



gain of T

$$\text{where } H(\omega) = \begin{cases} T, & |\omega| < \frac{\omega_s}{2} \\ 0, & \text{else} \end{cases}$$

T used for sampling $\omega_s = \frac{2\pi}{T}$, sampling frequency.

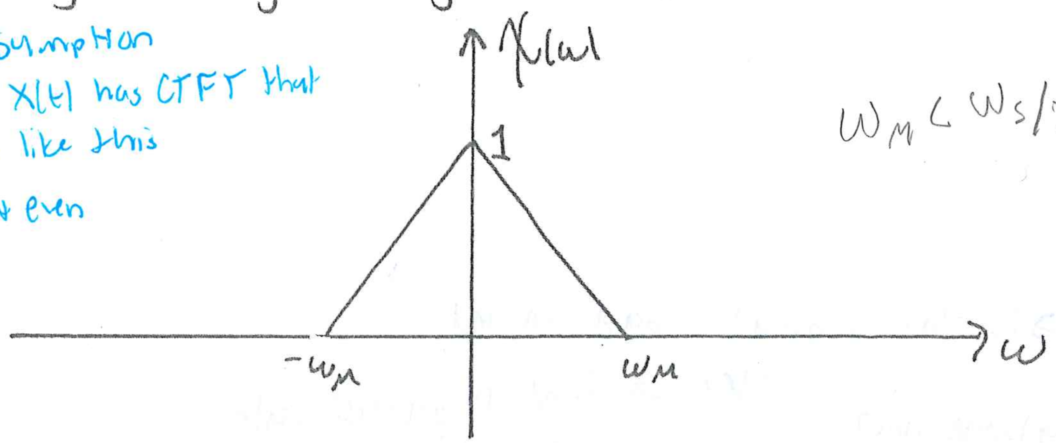
To understand why this work, we look at sampling and reconstruction in the frequency domain.

let's say the original signal looks like this:

→ assumption

Suppose x(t) has CTFT that looks like this

→ real + even



$\omega_M < \omega_s/2$

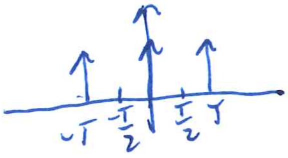
What is the CTFT of $P_T(t)$?

$$P(\omega) = \sum_{k=-\infty}^{\infty} a_k 2\pi \delta(\omega - \omega_s k)$$

where a_k are the F. series coeffs of $P_T(t)$

since

$$P_T(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\frac{2\pi}{T}t} = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_s t}$$



→ compute a_k's & bc easy (only 1 δ)

$$a_k = \frac{1}{T}\int_{-T/2}^{-T/2} P_T(t) e^{-jk\omega_s t} dt$$

$$= \frac{1}{T}\int_{-T/2}^{T/2} \delta(t) e^{-jk\omega_s t} dt = \frac{1}{T}$$

so...

$$\underbrace{P(\omega) = \frac{2\pi}{T}\sum_{k=-\infty}^{\infty} \delta(\omega - \omega_s k)}$$

$$X_{p(\omega)} = \mathcal{F}(X_p(t))$$

$$= \mathcal{F}(X(t)P_T(t))$$

$$= \frac{1}{2\pi} X(\omega) * P(\omega)$$

→ is this signal band limited

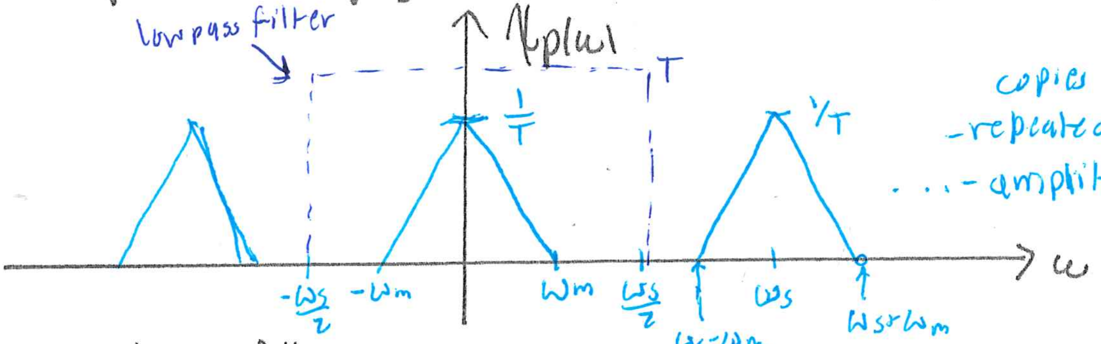
yes → find Nyquist rate

– justify answer

($\mathcal{F}$(signal)) → look at $\mathcal{F}$ domain

X(t) = sinc → band lim? yes - rect

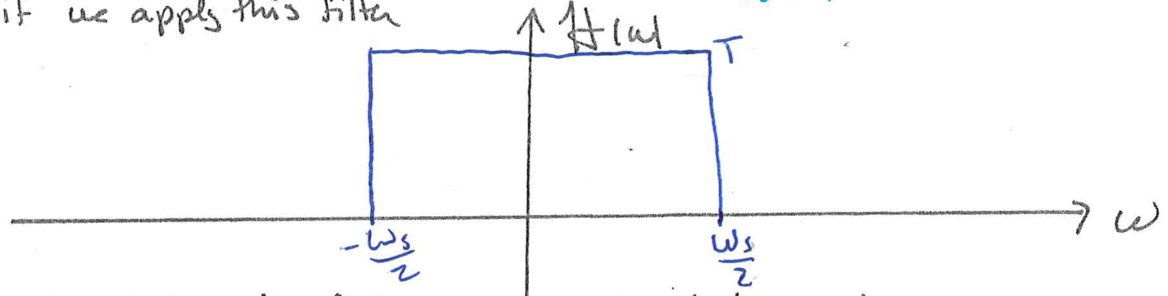
X(t) = rect → band lim? no - bc sinc in $\mathcal{F}$ (never stops) then the impulse train sampling will look like this: (*) proof later



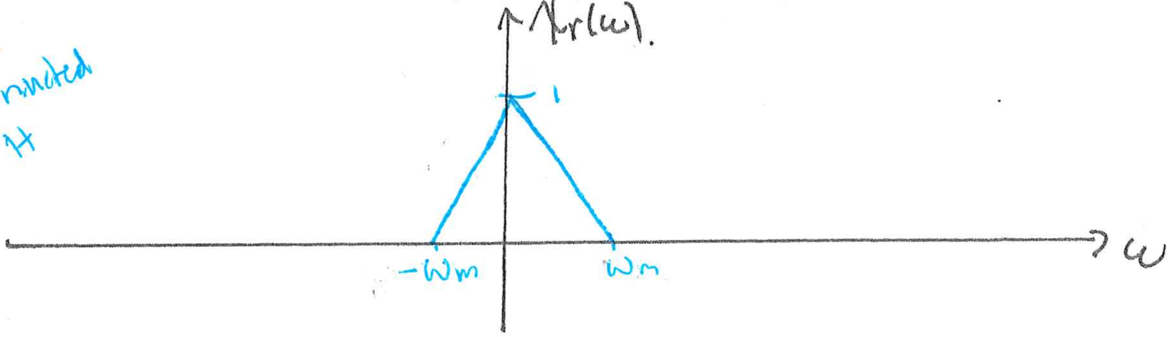
copies of $X(\omega)$
- repeated every ω_s
... - amplitude divided by T

multiply

So if we apply this filter



We will obtain the following reconstructed signal



reconstructed X

(*) Why is the CTFT of impulse-train sampling a repetition of $X(\omega)$?

because $X_p(\omega) = \mathcal{F}(\ x(t)\ p(t)\ )$

CTFT of $x_p(t)$ $\mathcal{F}(x_p(t)) =$

$$= \frac{1}{2\pi} X(\omega) * P(\omega).$$

but $P(\omega) = \sum_{k=-\infty}^{\infty} a_k\, 2\pi\, \delta(\omega - \omega_s k)$, with $a_k = \frac{1}{T}$

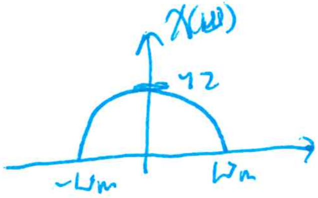
calculation

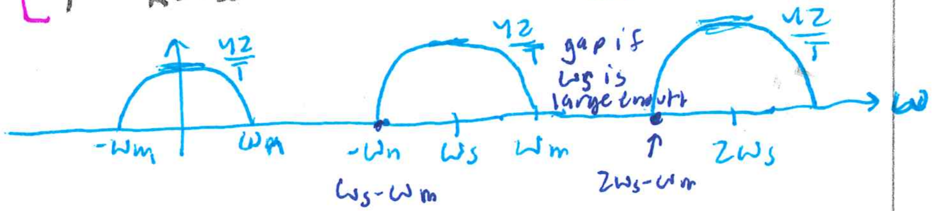
Therefore $X_p(\omega) = \frac{1}{2\pi} X(\omega) * P(\omega)$

$$= \frac{1}{\cancel{2\pi}} X(\omega) * \sum_{k=-\infty}^{\infty} \frac{\cancel{2\pi}}{T} \delta(\omega - \omega_s k)$$

$$= \sum_{k=-\infty}^{\infty} \frac{1}{T} X(\omega) * \delta(\omega - k\omega_s)$$

$$= \left[\frac{1}{T} \sum_{k=-\infty}^{\infty} X(\omega - \omega_s k) \right]$$ ✱ important





$$\overset{n=0}{\frac{1}{T} X(\omega)} + \overset{n=1}{\frac{1}{T} X(\omega - \omega_s)} + \overset{n=2}{\frac{1}{T} X(\omega - 2\omega_s)}$$

gap if $\omega_s - \omega_m > \omega_m$

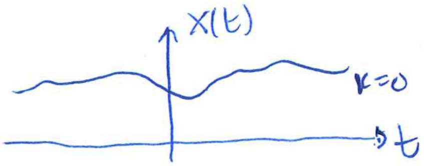
$\boxed{\omega_s > 2\omega_m}$

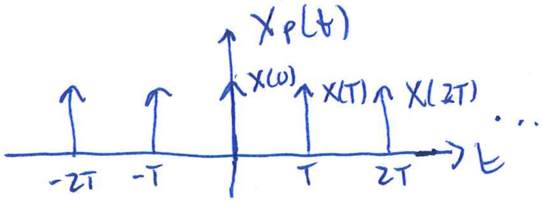
✱ Nyquist Condition

What happens in the time-domain?

"band-limited interpolation"

$$x_r(t) = \sum_{k=-\infty}^{\infty} x(kT) \frac{\sin(\omega_c(t-kT))}{t-kT}$$





Derivation of interpolation formula:

$$x_r(t) = x_p(t) \ast h(t)$$

$$= \left(\sum_{n=-\infty}^{\infty} \delta(t-nT)\, x(t) \right) \ast h(t)$$

$$= \left(\sum_{n=-\infty}^{\infty} \delta(t-nT)\, x(nT) \right) \ast h(t)$$

$$= \sum_{n=-\infty}^{\infty} x(nT) \quad \delta(t-nT) \ast h(t)$$

$$= \sum_{n=-\infty}^{\infty} x(nT) \quad h(t-nT)$$

but $h(t) = \frac{T \sin(\omega_c t)}{\pi t}$

ω_c = cut off requirement

$= \frac{1}{2} \omega_s$

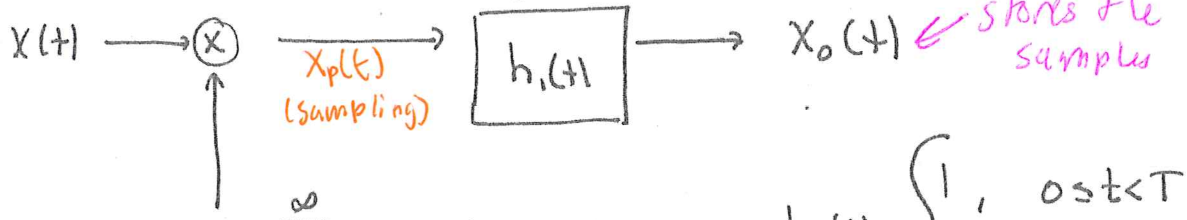
$$\Rightarrow x_r(t) = \sum_{n=-\infty}^{\infty} x(nT) \frac{T}{\pi} \frac{\sin(\frac{1}{2}\omega_s(t-nT))}{t-nT}$$

Part III - A

2.5

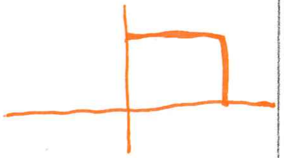
Another way to store the values of samples $x_d[n] = x(nT)$

" sampling with zero-order hold"



$$p_T(t) = \sum_{n=-\infty}^{\infty} \delta(t-kT)$$

$$h_1(t) = \begin{cases} 1, & 0 \le t < T \\ 0, & \text{else} \end{cases}$$



what does $x_0(t)$ look like?

$$x_0(t) = h_1(t) * (x_p(t))$$

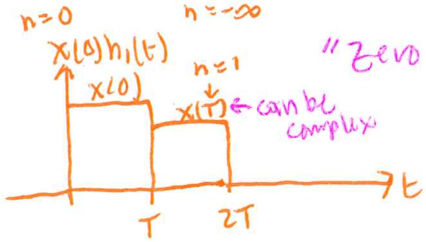
$$= h_1(t) * (x(t)\, p_T(t))$$

$$= h(t) * \left(x(t) \sum_{n=-\infty}^{\infty} \delta(t-nT)\right)$$

$$= h(t) * \sum_{n=-\infty}^{\infty} x(nT)\delta(t-nT)$$

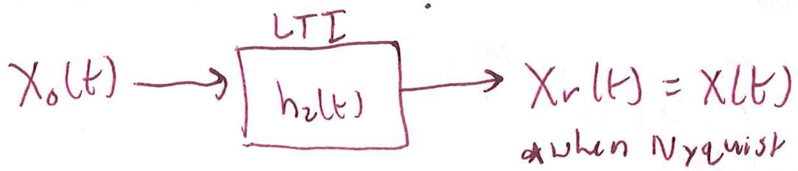
$$= \sum_{n=-\infty}^{\infty} x(nT)\; h(t)*\delta(t-nT) = \sum_{n=-\infty}^{\infty} x(nT)\, h_1(t-nT)$$

"zero-order hold"



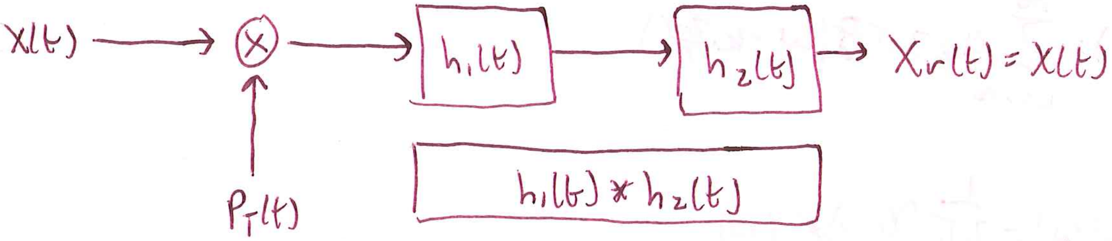
How to recover $x(t)$ from $x_0(t)$?

we want



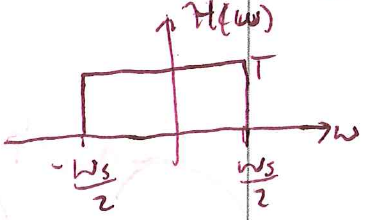
*when Nyquist condition is satisfied

i.e.



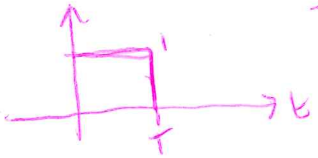
LPH → low pass filter
w/ gain T and cutoff $\frac{\omega_s}{2}$

i.e. want $\underbrace{H_1(\omega)}_{\text{known}} H_2(\omega) = \begin{cases} T, & |\omega| < \frac{\omega_s}{2} \\ 0, & \text{else} \end{cases} =: H(\omega) = \left[u\left(\omega + \frac{\omega_s}{2}\right) - u\left(\omega - \frac{\omega_s}{2}\right)\right]T$



bc $H_1(\omega) = \mathcal{F}(h_1(t))$

$= \frac{e^{-j\frac{\omega}{2}T} 2\sin\left(\frac{\omega}{2}T\right)}{\omega}$



so $\boxed{H_2(\omega) = \frac{H(\omega)}{H_1(\omega)}}$

So $x_r(t) = x(t)$ when Nyquist condition satisfied

$$x_p(t) = x(t)\,\underbrace{c(t)}_{\text{periodic}}$$

$$c(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\frac{2\pi}{T}t}$$

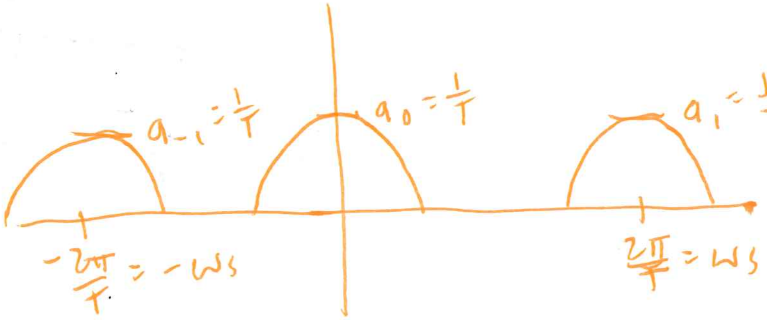
guess CTFT

$$C(\omega) = \sum_{k=-\infty}^{\infty} a_k 2\pi\delta\left(\omega - k\frac{2\pi}{T}\right)$$

$$X_p(\omega) = \frac{1}{2\pi} X(\omega) * C(\omega)$$

$$= \frac{1}{2\pi} X(\omega) * \sum_{k=-\infty}^{\infty} a_k 2\pi\delta\left(\omega - k\frac{2\pi}{T}\right)$$

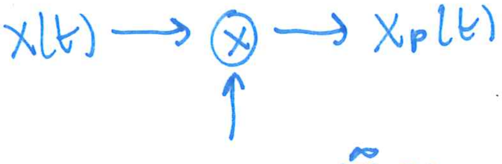
$$= \sum_{k=-\infty}^{\infty} X\left(\omega - k\frac{2\pi}{T}\right)$$



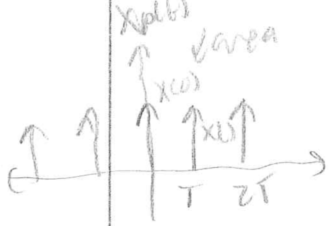
$a_k = \frac{1}{T}$, for all k

Summary

Impulse-train sampling

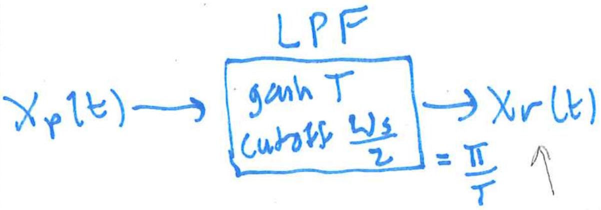


$$P_T(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT)$$



plot Xp(t) in $\mathcal{F}$ domain
→ just 1 in center
- copies of signal in $\mathcal{F}$ dom
w/

Reconstruction

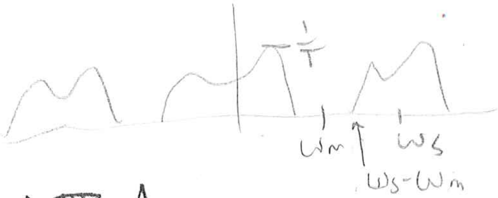


always band-limited

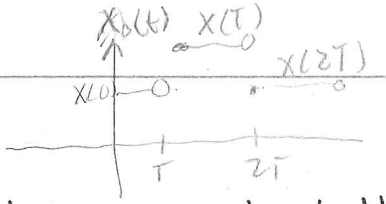
$x_r(t) = x(t)$

when $\omega_s > 2\omega_m$

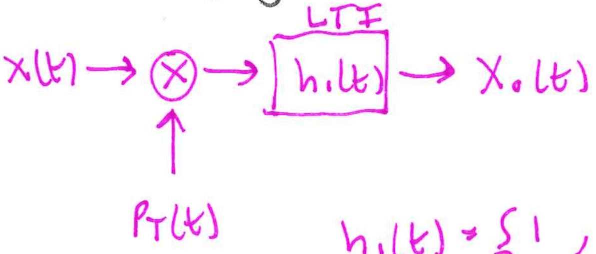
where ω_m s.t. $X(\omega)=0$ for $|\omega|>\omega_m$



[illegible] $\omega_s-\omega_m > \omega_m$
$\omega_s > 2\omega_m$



Sampling w/ zero-order hold



$$h_0(t) = \begin{cases} 1, & 0 \le t \le T \\ 0, & \text{else} \end{cases}$$

Reconstruction

$$H_2(\omega) = \frac{\omega\, H(\omega)}{e^{-j\omega t}\cdot 2\sin(\frac{\omega}{2}T)}$$

$x_0(t) = x(t)$

when $\omega_s > 2\omega_m$

3. Undersampling

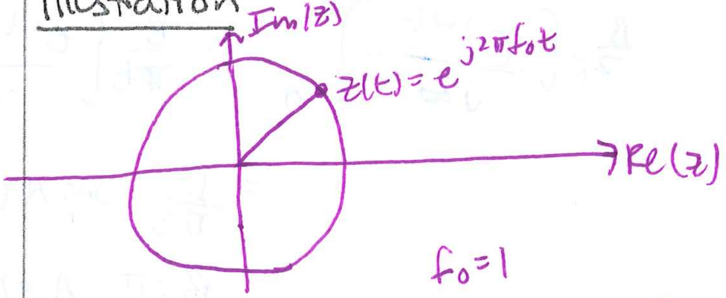
If $\omega_s \leq 2\omega_m$

What happens if sampling frequency is less than $2\omega_M$? (or equal to)

$\Rightarrow$ maybe not enough information to reconstruct $x(t)$.

"Aliasing" ← ✱ if overlap

Illustration

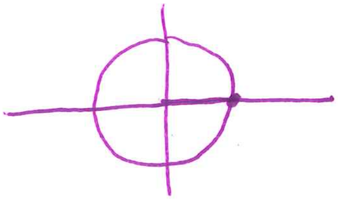


$f_0 = 1$

1Hz spinning freq

i.e. 1 turn per second

$t=1$ $z(1) = e^{j2\pi \cdot 1} = 1$



✱ just look @ $t=0$, $t=1$

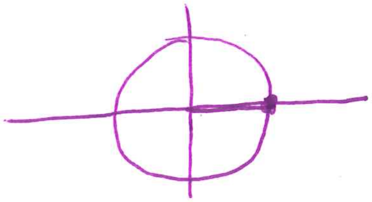
→ looks like not moving

$T = \frac{1}{4}$ $\omega_s = \frac{2\pi}{T} = 2\pi \cdot \underbrace{4}_{4\,Hz}$

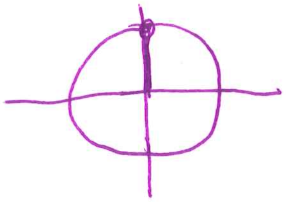
4 sample/sec

$t=0$

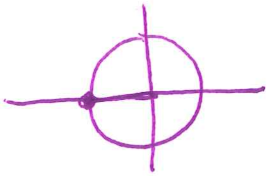
$z(0) = e^0 = 1$



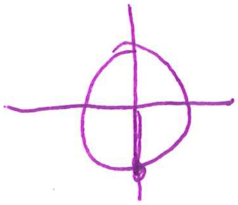
$t = \frac{1}{4}$ $z(\frac{1}{4}) = e^{j\frac{2\pi}{4}} = e^{j\frac{\pi}{2}} = j$



$t = \frac{1}{2}$ $z(\frac{1}{2}) = e^{j\frac{2\pi}{2}} = e^{j\pi} = -1$



$t = \frac{3}{4}$ $z(\frac{3}{4}) = e^{j2\pi \cdot \frac{3}{4}} = -j$



want

$$\mathcal{F}\left(\frac{\sin \pi t}{\pi t}\right) \longrightarrow X(\omega) = \begin{cases} B, & |\omega| < A \\ 0, & \text{els} \end{cases}$$

$$\mathcal{F}^{-1}(X(\omega)) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-A}^{A} B e^{j\omega t}\, d\omega = \frac{B}{2\pi}\left[\frac{e^{j\omega t}}{jt}\right]_{-A}^{A} = \frac{B}{\pi t}\left[\frac{e^{jt} - e^{-jt}}{2j}\right]$$

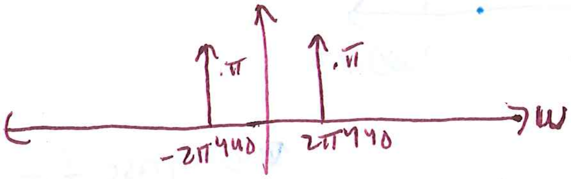
$$= \frac{B}{\pi t} \sin At$$

$$B = \pi \quad A = 1$$

A mathematical example of aliasing:

$x(t) = \cos(2\pi 440 t)$

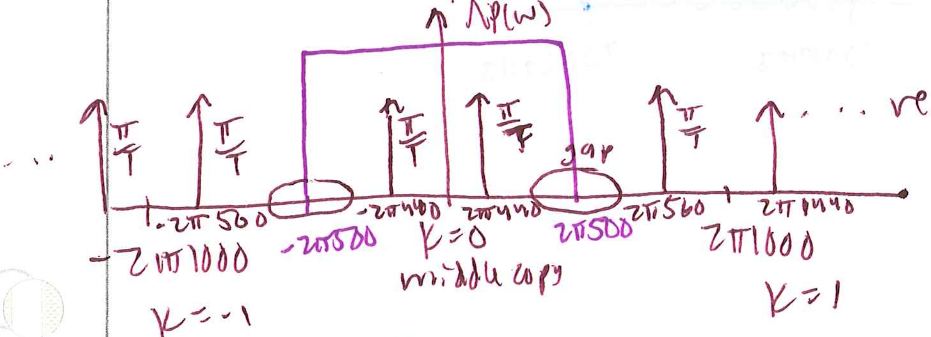
$X(\omega) = \pi\delta(\omega - 2\pi 440) + \pi\delta(\omega + 2\pi 440)$



WR = 2π 880

$x_p(t) = x(t)p_T(t)$, w $T = \frac{1}{1000}$

i.e. $\omega_s = \frac{2\pi}{T} = 2\pi \underbrace{1000}_{1000 Hz}$



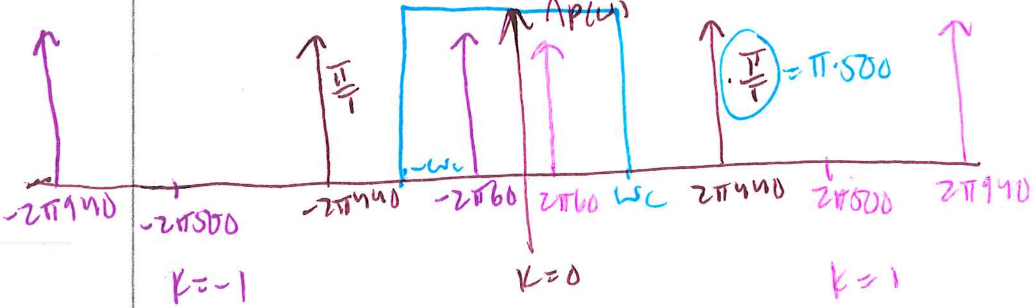
... repeat every ω_s (2π1000)

take $\omega_c = \frac{\omega_s}{2} = \underline{2\pi 500}$

gain = T

$X_p(\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X(\omega - k\omega_s)$

now take $T = \frac{1}{500}$ $\omega_s = \frac{2\pi}{T} = 2\pi \underbrace{500}_{500 Hz}$

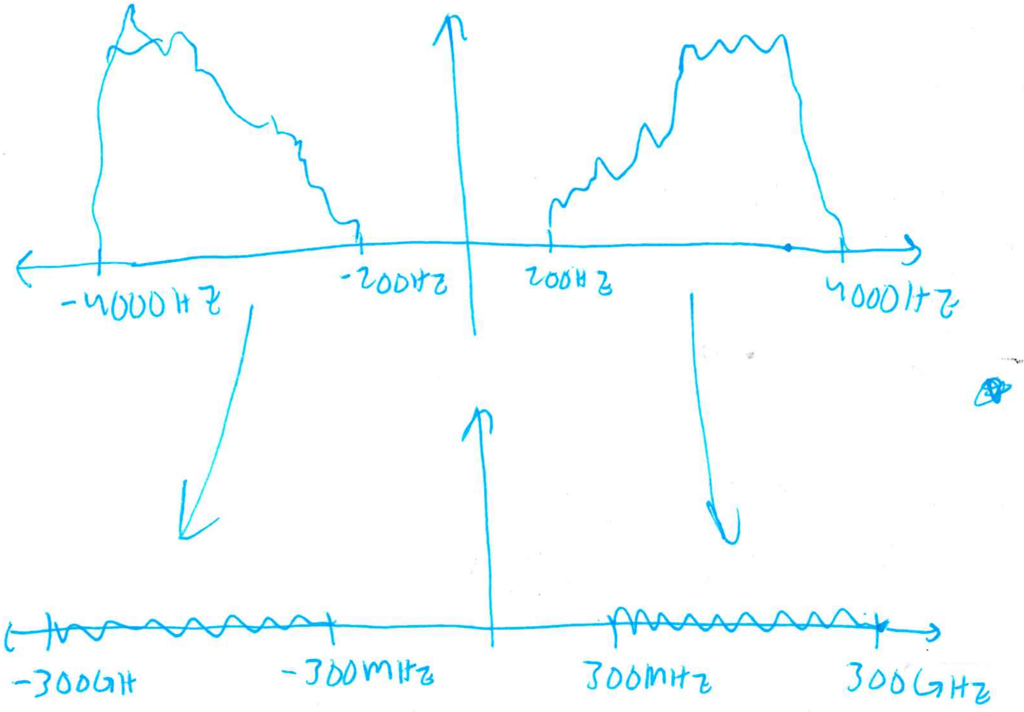


$\omega_c = \frac{\omega_s}{2} = \frac{2\pi 500}{2} = 2\pi 250$

$x_r(t) = \cos(2\pi 60 t)$

→ alias of og signal

$x(t) = \cos(2\pi 440 t)$

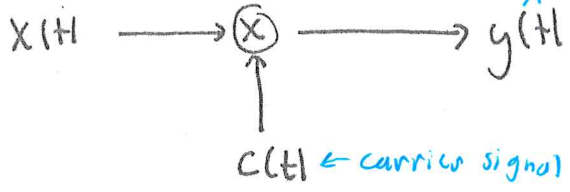


- change frequencies to be able to be sent thru channel

1. Amplitude Modulation with exponential and sine carriers

Modulator used to change frequencies of a signal.

Can do this using a "carrier signal" c(t).



*modulated signal

$y(t) = x(t)c(t)$

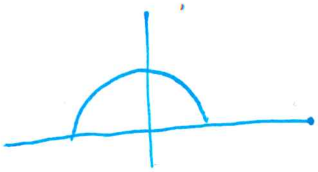
$Y(\omega) = \mathcal{F}(x(t)\,c(t))$

$= \frac{1}{2\pi} X(\omega) * C(\omega)$

Exponential carrier: $c(t) = e^{j\omega_c t}$

$$y(t) = e^{j\omega_c t}\, x(t)$$

$$\Rightarrow Y(\omega) = \mathcal{F}(x(t)\,c(t)) = \frac{1}{2\pi} X(\omega) * C(\omega)$$



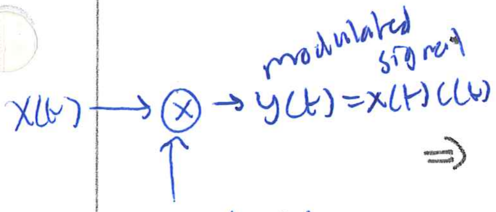
$$= \frac{1}{2\pi} X(\omega) * 2\pi\,\delta(\omega - \omega_c)$$

$$= X(\omega - \omega_c)$$

if $\omega_c > \omega_n$
no negative frequencies left in the signal

ex: $\cos(2\pi 440 t)$

Sine carrier: $c(t) = \cos(\omega_c t)$



$c(t) = \cos(\omega_0 t)$

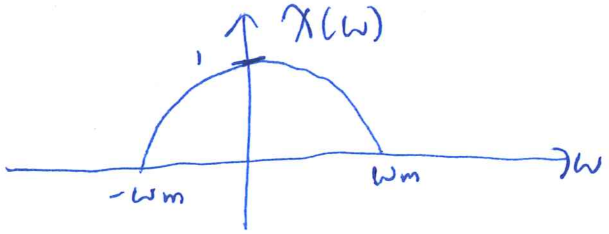
$= \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2}$

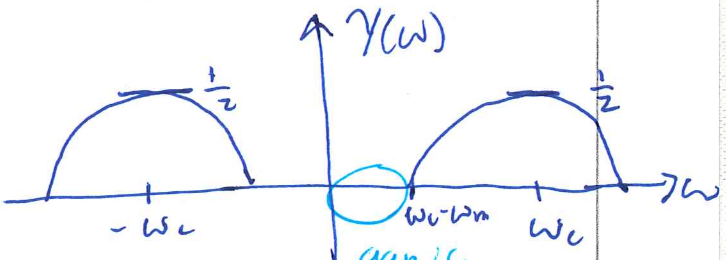
$y(t) = \cos(\omega_c t)\, x(t)$

$$\Rightarrow Y(\omega) = \mathcal{F}(\cos(\omega_0 t)\, X(t)) = \frac{1}{2\pi} \mathcal{F}(\cos(\omega_0 t)) * X(\omega)$$

$$= \frac{1}{2\pi}\left[\pi \delta(\omega - \omega_0) + \pi \delta(\omega + \omega_0)\right] * X(\omega)$$

$$= \frac{X(\omega - \omega_c) + X(\omega + \omega_c)}{2}$$





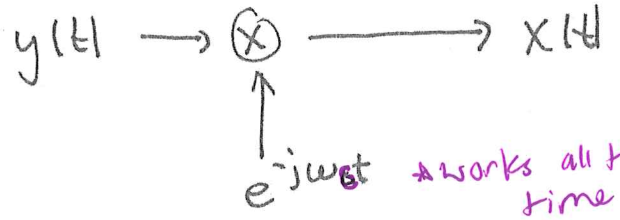
gap if $\omega_c > \omega_m$

$\omega_c > \omega_m$

Part III B 1.2

2. Demodulation

How to demodulate $y(t) = e^{j\omega_c t} x(t)$?



*works all the time

*any ω_c value

In frequency domain:

$$\mathcal{F}\left(e^{-j\omega_c t} y(t)\right) = \frac{1}{2\pi} \mathcal{F}\left(e^{-j\omega_c t}\right) * Y(\omega)$$

$$= \frac{1}{2\pi} 2\pi \delta(\omega + \omega_c) * Y(\omega)$$

$$= Y(\omega + \omega_c) \quad \text{but } Y(\omega) = X(\omega - \omega_c)$$

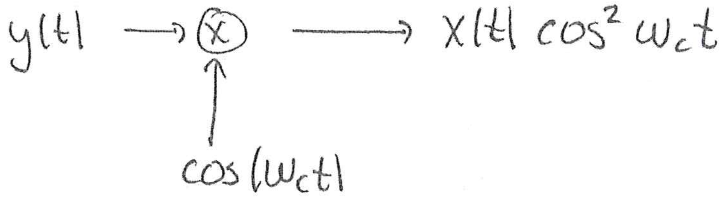
$$= X(\omega) = X(\omega - \omega_c + \omega_c)$$

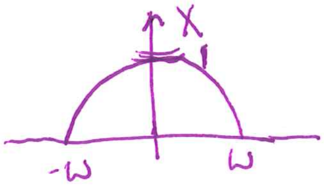
How to demodulate $y(t) = x(t)\cos(\omega_c t)$?

Assume $X(\omega) = 0$ for $|\omega| > \omega_m$

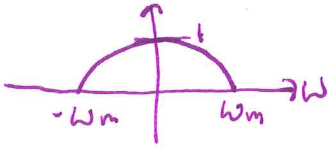
and $\omega_c > \omega_m$

Step 1

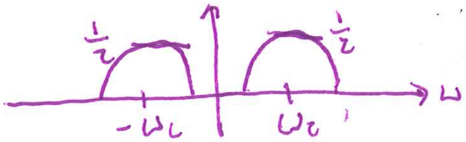




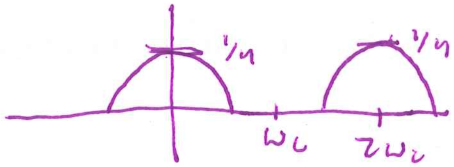
$X(t) \xrightarrow{CTFT}$



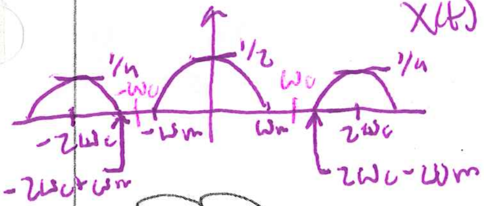
$X(t)\cos(\omega_c t) \xrightarrow{CTFT}$



$X(t)\cos^2(\omega_c t) \xrightarrow{CTFT}$



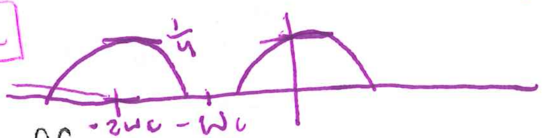
"right side copy"



gap if $\omega_m < 2\omega_c - \omega_m$

$\boxed{\omega_m < \omega_c}$

"leftside copy"



Step 2 LPF, gain 2, cutoff ω_{LPF}

with $\boxed{\omega_M < \omega_{LPF} < 2\omega_c - \omega_m}$ ✱

Will work if $\omega_m < \omega_c$

✱ assume X(t) is band-limited

✱ LHS copy, RHS copy

✱ (add)

✱ LPF to isolate middle copy

Part III B 2.2

Q's

modulate under ___ modulation

What condition must be satisfied.

Cos
- → band limited
- → ω_c ?

expo
- → no condition

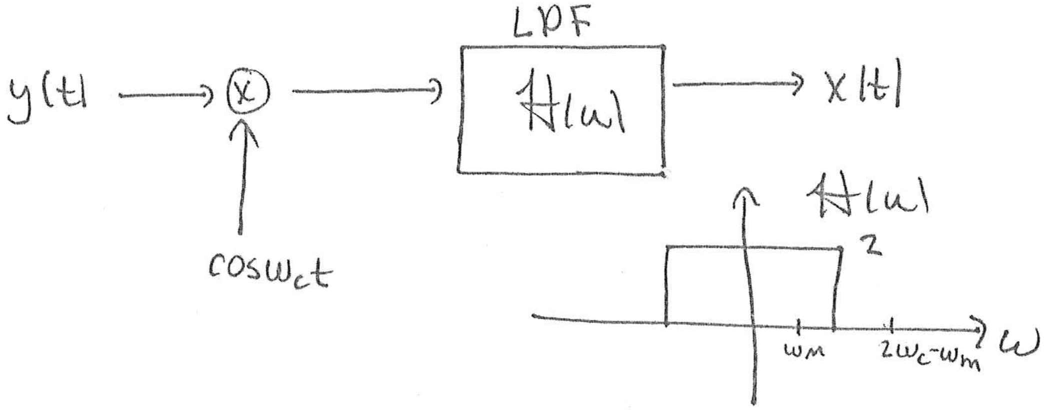
in time-domain:

$$x(t)\cos^2 \omega_c t = x(t)\left[\frac{1}{2} + \frac{1}{2}\cos 2\omega_c t\right]$$

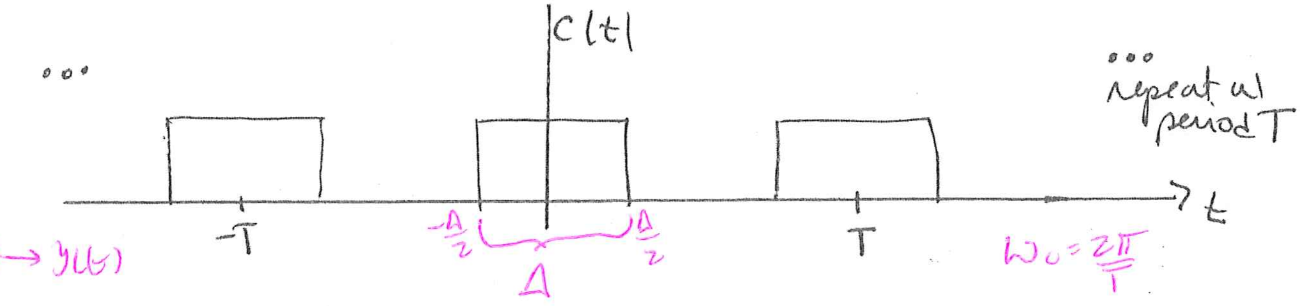
$$= \underbrace{\frac{1}{2}x(t)}_{\text{signal in middle of spectrum}} + \underbrace{\frac{1}{2}x(t)\cos 2\omega_c t}_{\text{signal left + right of the middle signal}}$$

two opposite copies at ± ω_c

°° Demodulation



3. Amplitude modulation with a pulse-train carrier



$\omega_0 = \frac{2\pi}{T}$

$$\begin{tikzcd} x(t) \arrow[r] & \otimes \arrow[r] & y(t) \\ & c(t) \arrow[u] & \end{tikzcd}$$

$$y(t) = x(t)\,c(t)$$

$Y(\omega) = \mathcal{F}(x(t)\,c(t))$

$$\Rightarrow \; Y(\omega) = \frac{1}{2\pi} X(\omega) * C(\omega).$$

works all the time for periodic sig

$* \; c(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\frac{2\pi}{T}t}$

$$C(\omega) = \sum_{k=-\infty}^{\infty} a_k \, 2\pi \, \delta(\omega - \tfrac{2\pi}{T}k)$$

$* \; c(t) = \sum_{k=-\infty}^{\infty} u(t+\frac{\Delta}{2} + kT) - u(t - \frac{\Delta}{2} + kT)$

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} c(t)\, e^{-jk\frac{2\pi}{T}t}\, dt$$

$$= \frac{1}{T}\int_{-\Delta/2}^{\Delta/2} 1 \; e^{-jk\frac{2\pi}{T}t}\, dt$$

$$= \frac{1}{T} \left. \frac{e^{-jk\frac{2\pi}{T}t}}{-jk\frac{2\pi}{T}} \right|_{-\frac{\Delta}{2}}^{\frac{\Delta}{2}}$$

if $k = 0$

$$a_0 = \frac{\Delta}{T}$$

$* \; a_0$ is always avg of signal over 1 period b-a

if $k \neq 0$

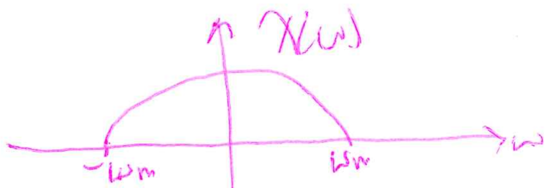
$$a_k = \quad = \frac{1}{\pi k} \sin\left(\frac{\pi}{T} k \Delta\right)$$

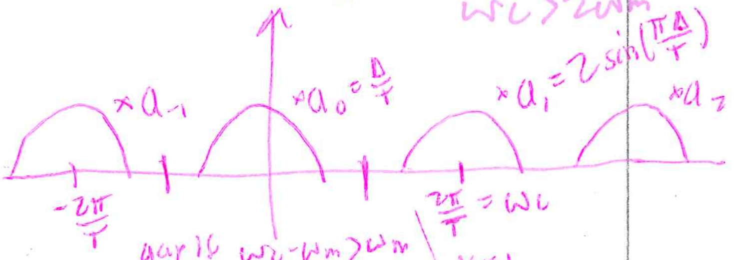
so $Y(\omega) = \frac{1}{2\pi} X(\omega) * \sum a_k 2\pi \delta(\omega - k\frac{2\pi}{T})$

$= \sum_{k=-\infty}^{\infty} a_k X(\omega) * \delta(\omega - k\frac{2\pi}{T}) = \sum_{k=-\infty}^{\infty} a_k X(\omega - k\frac{2\pi}{T})$

$* \; \omega_c - \omega_m > \omega_m$

$\omega_c > 2\omega_m$





gap is $\omega_c - \omega_m > \omega_m$

$\omega_c > 2\omega_m$

k=1

$\frac{\Delta}{T}$ not 2π

$w_0 \Rightarrow \frac{2\pi}{T}$ or w_c

So

$$C(\omega) = \frac{\Delta}{T} 2\pi \delta(\omega) + \sum_{k \neq 0} \frac{1}{k\pi} \sin\left(k \frac{\omega_0 \Delta}{2}\right) 2\pi \, \delta(\omega - k\omega_0)$$

$$= \frac{2\pi}{T} \Delta \, \delta(\omega) + \sum_{k \neq 0} \frac{2}{k} \sin\left(k \frac{\omega_0 \Delta}{2}\right) \delta(\omega - k\omega_0)$$

Then

$$Y(\omega) = \frac{1}{2\pi} X(\omega) * C(\omega)$$

$$= \frac{1}{2\pi} X(\omega) * \sum_{k=-\infty}^{\infty} a_k \, 2\pi \, \delta(\omega - k\omega_0)$$

$$= \sum_{k=-\infty}^{\infty} a_k \, X(\omega) * \delta(\omega - k\omega_0)$$

$$= \sum_{k=-\infty}^{\infty} a_k \, X(\omega - k\omega_0)$$

$$= \frac{2\pi}{T} \Delta \, X(\omega) + \sum_{k \neq 0} \frac{2}{k} \sin\left(k \frac{\omega_0 \Delta}{2}\right) X(\omega - k\omega_0)$$

How to demodulate? w/

✱ Not on exam 3 - only final!

1. The Laplace Transform

Recall
$$e^{st} \longrightarrow \boxed{\text{LTI}} \longrightarrow \overbrace{H(s)}^{\text{number}}\, e^{st}$$

where
$$H(s) = \int_{-\infty}^{\infty} h(t)\, e^{-st}\, dt \qquad \text{"Transfer Funct"}$$

h(t) = unit impulse response of system

Define:

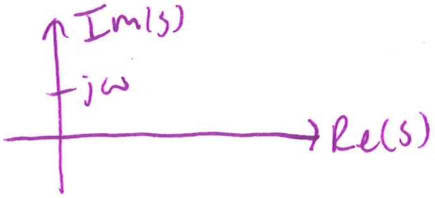
$$\boxed{\begin{array}{l}\text{The "Laplace transform" of a signal } x(t) \text{ is} \\ \underset{\substack{\uparrow \\ \text{complex num}}}{X(s)} = \displaystyle\int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt. \;, \qquad s \in \mathbb{C}\end{array}}$$

What is the difference between Laplace transform $X(s)$ and Fourier transform $\mathcal{X}(\omega)$?

$$\underset{\text{complex}}{\underset{\uparrow}{X(s)}} \qquad\qquad \underset{\uparrow}{\mathcal{X}(\omega)}$$

Observe $$X(s)\Big|_{s=j\omega} = \int_{-\infty}^{\infty} x(t)\, e^{-st}\, dt \,\Bigg|_{s=j\omega}$$

$$= \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$



$$= \mathcal{X}(\omega)$$

The Fourier transform of a signal is the restriction of the ~~Fourier~~ Laplace transform of that signal to the imaginary axis in the complex plane.

*know how to do!

Examples of computation of Laplace transforms.

i) $x(t) = e^{-at} u(t)$, a real, $a > 0$

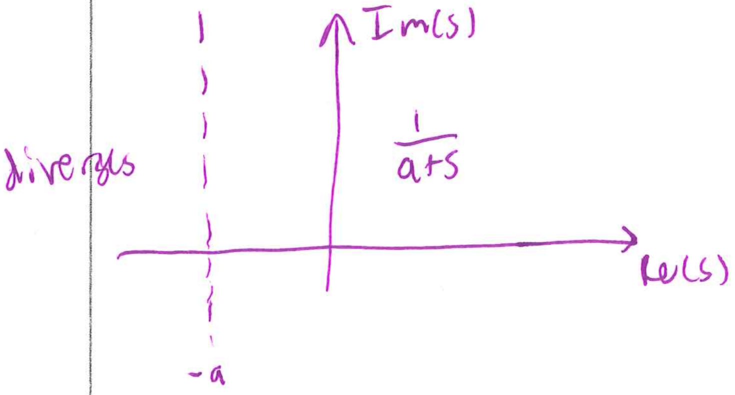
$$X(s) = \int_{-\infty}^{\infty} e^{-at} u(t) e^{-st} \, dt = \int_{0}^{\infty} e^{-(a+s)t} \, dt$$

$$= \frac{e^{-(a+s)t}}{-(a+s)} \Bigg|_{0}^{\infty}$$

$$e^{-(a+s)t} = e^{-\overbrace{Re(a+s)t}^{(+)} - \underbrace{jIm(a+s)t}_{\text{oscillates}}}$$

$$= \begin{cases} 0 - \frac{e^{0}}{-(a+s)} , \text{if } Re(a+s) > 0 \\ \text{diverges, else} \end{cases}$$

$$X(s) = \begin{cases} \frac{1}{a+s} , \text{if } Re(s) > -a \\ \text{diverges, else} \end{cases}$$



write

$$X(s) = \frac{1}{a+s} , \underset{\uparrow}{ROC} \; Re(s) > -a$$

"region of convergence"

→ FT replace s w/ $j\omega$ $\quad X(\omega) = \frac{1}{a+j\omega}$

2) $x(t) = -e^{-at} u(-t)$, a real, $a > 0$

$$X(s) = \int_{-\infty}^{\infty} -e^{-at} u(-t) e^{-st} \, dt = \int_{-\infty}^{0} -e^{-(a+s)t} \, dt$$

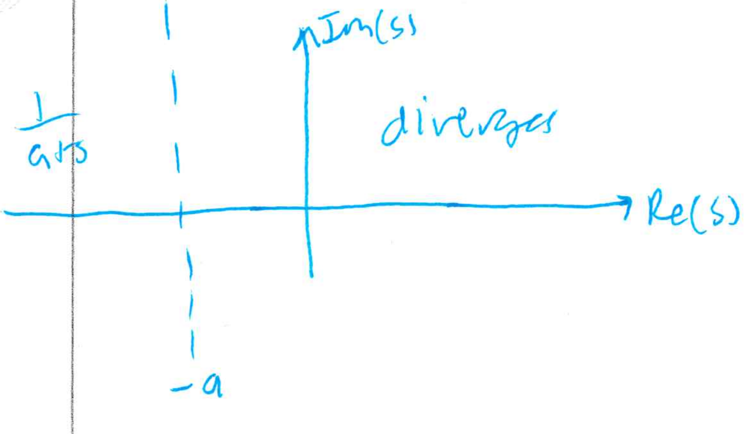
$$= \frac{-e^{-(a+s)t}}{-(a+s)} \Bigg|_{-\infty}^{0}$$

$*\ e^{-(a+s)t} = e^{\overbrace{-Re(a+s)t}^{\text{want (+) here}}} e^{-jIm(a+s)t}$

Want: $Re(a+s) < 0$ for convergence

$Re(s) < -a$

$$= \begin{cases} \frac{1}{a+s}, & Re(s) < -a \\ \text{diverges}, & \text{else} \end{cases}$$



*FT would converge

* ~~same~~ similar answer as ex 1 but diff Laplace Transform

$< -a$ / $> -a$

Summary

$x(t)$	$X(s)$	converges for
$e^{-at}u(t)$	$\frac{1}{a+s}$	$\mathrm{Re}(s) > -a$
$-e^{-at}u(-t)$	$\frac{1}{a+s}$	$\mathrm{Re}(s) < -a$

"ROC" = region of convergence

2. Laplace transform and LTI systems

$$H(s) = \int_{-\infty}^{\infty} h(t) e^{-st} dt$$

" transfer function of the system "

$H(j\omega) = \mathcal{H}(\omega)$ " frequency response of the system "

The properties of the transfer function (esp. ROC) correspond to properties of the system.

Example: An LTI system with a finite duration $h(t)$ has a transfer function that converges on the entire complex plane.

Lemma: If $x(t)$ is of finite duration

i.e. $\exists\, t_M$ s.t. $x(t) = 0$ for $|t| > t_M$

and if $\int_{-\infty}^{\infty} |x(t)| dt$ is finite

then $X(s) = \int_{-\infty}^{\infty} x(t) e^{-st} dt$ converges for all values of $s \in \mathbb{C}$.

proof:
$$\left| \int_{-\infty}^{\infty} x(t) e^{-st} dt \right| = \left| \int_{-t_M}^{t_M} x(t) e^{-st} dt \right|$$
$$\leq \int_{-t_M}^{t_M} |x(t) e^{-st}| dt$$
$$= \int_{-t_M}^{t_M} |x(t)| e^{-Re(s)t} dt$$
$$\leq e^{|Re(s)| t_M} \int_{-t_M}^{t_M} |x(t)| dt < \infty.$$

3. The Z transform (~DT Laplace Transform)

Recall

where $H(z) = \sum_{n=-\infty}^{\infty} h[n] z^{-n}$.

$h[n]$ unit impulse response of system

$H(z)$ = "transfer function of system"

Define:

$$\boxed{\text{The "Z transform" of a DT signal } x[n] \text{ is} \quad X(z) = \sum_{n=-\infty}^{\infty} \cancel{X}[n] z^{-n}, \quad z \in \mathbb{C}}$$

z complex numbers

$H(z)$ —— DT

$H(s)$ —— CT

What is the difference between the Z transform $X(z)$ and the Fourier transform $\mathcal{X}(\omega)$?

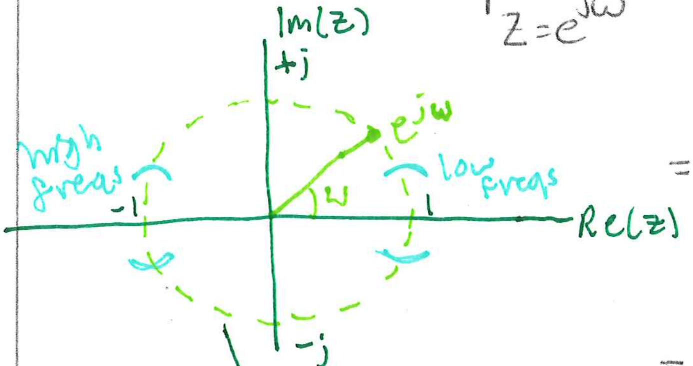
$$\underset{\substack{\uparrow \\ \text{complex}}}{X(z)} \qquad\qquad \underset{\substack{\uparrow \\ \text{real}}}{\mathcal{X}(\omega)}$$

Observe

$$X(z)\Big|_{z=e^{j\omega}} = \sum_{n=-\infty}^{\infty} x[n]\, z^{-n}\Big|_{z=e^{j\omega}}$$

$$= \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

$$= \mathcal{X}(\omega)$$



$$\mathcal{X}(\omega) = X(e^{j\omega})$$

The ~~Z-transform~~ Fourier Transform DTFT of a signal is the restriction of the z-transform of that signal on the unit circle in complex plane.

The z-transform of a signal may diverge for some values of z.

Definition: The set of $z \in \mathbb{C}$ at which

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} \quad \text{converges}$$

is called the "Region of Convergence" of $X(z)$.

Example 1 $x[n] = 2^{-n} u[n]$

$$X(z) = \sum_{n=-\infty}^{\infty} 2^{-n} u[n] z^{-n}$$

$$= \sum_{n=0}^{\infty} 2^{-n} z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{2z}\right)^n$$

$$= \begin{cases} \frac{1}{1-\frac{1}{2z}}, & \text{if } \left|\frac{1}{2z}\right| < 1 \\ \text{diverges}, & \text{else} \end{cases}$$

$$= \begin{cases} \frac{1}{1-\frac{1}{2}z^{-1}}, & \text{if } |z| > \frac{1}{2} \\ \text{diverges}, & \text{else} \end{cases}$$

$$X(\omega) = X(e^{j\omega})$$

$$= \frac{1}{1-\frac{1}{2}e^{-j\omega}} \longleftarrow \text{now have FT}$$

Example 2 $\quad x[n] = -2^{-n}\, u[-n-1]$

$$X(z) = \sum_{n=-\infty}^{\infty} -2^{-n}\, u[-n-1]\, z^{-n}$$

$$= \sum_{n=-\infty}^{-1} -2^{-n}\, z^{-n}$$

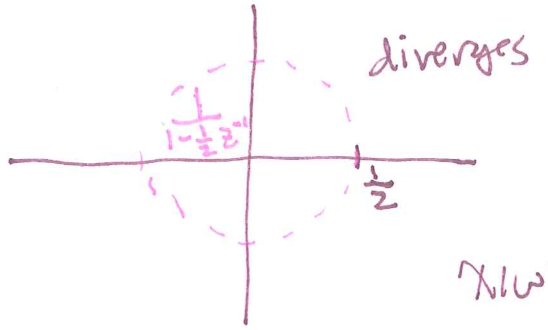
let $m = -n-1$

$$= \sum_{m=0}^{\infty} -2^{m+1}\, z^{m+1}$$

$$= -2z \sum_{m=0}^{\infty} (2z)^m$$

$$= -2z \begin{cases} \dfrac{1}{1-2z}, & \text{if } |2z| < 1 \\ \text{diverges,} & \text{else} \end{cases}$$

$$= \begin{cases} \dfrac{1}{1-\frac{1}{2}z^{-1}}, & \text{if } |z| < \frac{1}{2} \\ \text{diverges,} & \text{else} \end{cases}$$



$X(\omega)$ diverges

4. Z transform and LTI systems

$$H(z) = \sum_{n=-\infty}^{\infty} h[n] z^{-n}$$

"transfer function of the system"

$$H(e^{j\omega}) = \mathcal{H}(\omega) = \text{" frequency response of system"}$$

The properties of the transfer function (esp. ROC) correspond to properties of the system.

Example: An LTI system with a finite duration $h[n]$ has a transfer function that converges on the entire complex plane, except perhaps the origin $z=0$.

Lemma: If $x[n]$ is of finite duration

i.e. $\exists\ n_M$ s.t. $x[n]=0$ for $|n|>n_M$

and if $x[n] < \infty$ for all n

then $X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$ converges for all $z \neq 0$.