# Mock Qual #1

*Instructions: students should take this under exam conditions, i.e. no books, notes, etc. Upon completion, each student is encouraged to submit his/her exam for grading (although this is optional). The solutions will be evaluated for mathematical correctness and suitability for the qual setting, and returned next class.*

1. Let $f$ be a holomorphic mapping from the unit disc to itself. Show that, for all $|a| < 1$,
$$\frac{|f'(a)|}{1-|f(a)|^2} \leq \frac{1}{1-|a|^2}.$$
What can be said if equality holds?

2. Evaluate
$$\int_0^\infty \frac{\sin x}{x}\, dx$$
by complex variable methods.

3. Suppose that $f \in \mathcal{O}(\Omega - \{a_1, a_2\})$, where $\Omega$ is a simply connected domain and $a_1, a_2 \in \Omega$. Denoting $R_i := \operatorname{Res}_{z=a_i} f$ the respective residues, show that there exists an analytic function $F(z)$ on $\Omega - \{a_1, a_2\}$ satisfying
$$F'(z) = f(z) - \frac{R_1}{z-a_1} - \frac{R_2}{z-a_2}.$$

4. Let $\gamma$ be the ellipse $9x^2 + 16y^2 = 144$, with its positive orientation. Define
$$\varphi(z) = \int_\gamma \frac{e^{2|\zeta|}}{\zeta - z}\, d\zeta.$$
   (a) Show that $\varphi$ is analytic in a neighborhood of zero.
   (b) Find the radius of convergence for the Taylor series of $\varphi$ at $z = 0$. [Hint: the direct way might not be the best here.]
   (c) Estimate $|\varphi''(0)|$ from above.

5. Let $f$ be analytic in a domain containing the closed disk $\{z : |z| \leq 2\}$, and let $C$ denote its boundary, traversed once counterclockwise. Prove that
$$\operatorname{Re} f(0) = \frac{1}{2\pi i}\int_C \frac{\operatorname{Re} f(z)}{z}\, dz.$$