10

(15 pts) **1.** Using the definition of the Fourier transform (*not* the table of Fourier transform pairs), compute the Fourier transform of the DT signal:

$$x[n] = \left(\frac{1}{2j}\right)^{|n|}.$$

-1

$$x[n] = \begin{cases} \left(\frac{1}{2j}\right)^{n} & n \geq 0 \\ \left(\frac{1}{2j}\right)^{-n} & n < 0 \end{cases} = \begin{cases} \left(\frac{1}{2j}\right)^{n} u[n] \\ \left(\frac{1}{2j}\right)^{-n} u[-n+1] \end{cases}$$

$$x[n] = \left(\frac{1}{2j}\right)^{n} u[n] + \left(\frac{1}{2j}\right)^{-n} u[-n+1]$$

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} \left(\frac{1}{2j}\right)^{n} u[n] e^{-j\omega n} + \sum_{n=-\infty}^{\infty \; -1} \left(\frac{1}{2j}\right)^{-n} u[n+1] e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{2j} e^{-j\omega}\right)^{n} + \sum_{n=-\infty}^{-1} \left(2j e^{-j\omega}\right)^{n}$$

$$\sum_{n=-1}^{-\infty} (\;)^{n} = \sum_{n=1}^{\infty} \frac{1}{(\;)^{n}}$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{2j} e^{-j\omega}\right)^{n} + \sum_{n=-1}^{-\infty} \left(2j e^{-j\omega}\right)^{n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{2j} e^{-j\omega}\right)^{n} + \frac{1}{\sum_{n=1}^{\infty} \left(2j e^{-j\omega}\right)^{n}}$$

for $\sum_{n=1}^{\infty} (2je^{-j\omega})^{n}$

let $k = n-1$ $\quad n = k+1$

$$\sum_{k=0}^{\infty} (2je^{-j\omega})^{k+1} = \sum_{k=0}^{\infty} (2je^{-j\omega})($$

$$2je^{-j\omega} \sum_{k=0}^{\infty} (2je^{-j\omega})^{k}$$

$$= 2je^{-j\omega} \left(\frac{1}{1-2je^{-j\omega}}\right)$$

Substitute back

$$= \frac{1}{1-\frac{1}{2j}e^{-j\omega}} + \frac{1}{\left[2je^{-j\omega}\left(\frac{1}{1-2je^{-j\omega}}\right)\right]}$$

12

(20 pts) **2.** The Frequency response of a continuous-time LTI system is

$$H(j\omega) = \mathcal{H}(\omega) = \frac{1}{j\omega + 2}.$$

Use the ~~convolution property~~ of the Fourier transform to determine the response $y(t)$ when the input is $x(t) = e^{-|t|}$. $y(t) = x(t) * h(t) \xrightarrow{\mathcal{F}} X(\omega)H(\omega)$ ✓

$$x(t) = e^{-|t|} = \begin{cases} e^{-t} & t \geq 0 \\ e^{t} & t < 0 \end{cases} = \begin{cases} e^{-t}u(t) \\ e^{t}\,\cancel{u(-t+1)} \end{cases}$$

shifting $u(t)$ in CT

$$x(t) = e^{-t}u(t) + e^{t}u(-t+1)$$

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$$

$$= \int_{-\infty}^{\infty} e^{-t}u(t)e^{-j\omega t}\,dt + \int_{-\infty}^{\infty} e^{-t}u(-t+1)e^{-j\omega t}\,dt$$

$$= \int_{0}^{\infty} e^{t(-1-j\omega)}\,dt + \int_{-\infty}^{-1} e^{t(-1-j\omega)}\,dt$$

$$= \frac{e^{t(-1-j\omega)}}{-1-j\omega}\Bigg]_{0}^{\infty} + \frac{e^{t(-1-j\omega)}}{-1-j\omega}\Bigg]_{-\infty}^{-1}$$

$$= \frac{1}{1+j\omega} + \frac{e^{1+j\omega}}{-1-j\omega} = \frac{1-e^{1+j\omega}}{1+j\omega}$$

$$\mathcal{F}(h(t) * x(t)) \longrightarrow X(\omega)H(\omega) = \frac{1-e^{1+j\omega}}{1+j\omega}$$

$$Y(\omega) = \frac{1-e^{1+j\omega}}{1+j\omega} = \frac{1}{1+j\omega} - e^{1+j\omega}\left(\frac{1}{1+j\omega}\right)$$

$$Y(\omega) = {}^{3}\frac{1}{1+j\omega} - e \cdot e^{j\omega}\left(\frac{1}{1+j\omega}\right)$$

$\downarrow$ property 7 $\qquad\qquad \downarrow$ property 10

$$y(t) = e^{-t}u(t) - e \cdot e^{-(t+1)}u(t+1)$$

-2 because of ... -2 ... time-shift ...

5

(15 pts) **3.** True/False? The Fourier transform of a DT signal $x[n]$ is a periodic function, no matter what $x[n]$ is. (Justify your answer.)

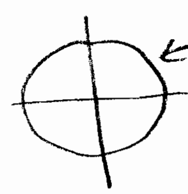
TRUE   Yes, The Fourier Transform of any DT signal is always periodic with period 2π

let $x[n]$

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

Let $x[n] = 1$

$$X(\omega) = \sum_{n=-\infty}^{\infty} e^{-j\omega n}$$



$\leftarrow e^{-j\omega n}$

$e^{-j\omega n}$ is a periodic signal, and it has period of 2π

when you take the Fourier transform of an input $x[n]$, you multiply it by $e^{-j\omega n}$, a periodic function and sum ~~the~~ result over a period of time. That must give you a periodic result/function

10

(10 pts) **4.** A continuous-time LTI system has frequency response

$$H(j\omega) = \frac{j\omega + 4}{(j\omega + 2)(j\omega + 3)}.$$

Derive a differential equation representing this system. (Use the properties of the Fourier transform listed in the table to justify your answer.)

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{j\omega + 4}{(j\omega+2)(j\omega+3)} = \frac{j\omega+4}{(j\omega)^2 + 5j\omega + 6}$$

$$\left[(j\omega)^2 + 5j\omega + 6\right] Y(\omega) = (j\omega + 4)\, X(\omega)$$

$$(j\omega)^2 Y(\omega) + 5j\omega\, Y(\omega) + 6\, Y(\omega) = j\omega\, X(\omega) + 4X(\omega)$$

$\downarrow F^{-1}$ $\qquad\qquad\qquad\qquad$ $\downarrow F^{-1}$

& by equation 16
Property

$$\frac{d^2}{dt} y(t) + 5\frac{d}{dt} y(t) + 6y(t) = \frac{d}{dt} x(t) + 4x(t)$$

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$$x[n] = \left(\frac{1}{2j}\right)^{n} u[n] + \left(\frac{1}{2j}\right)^{-n} u[-n+1]$$

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

$$= \sum_{n=-\infty}^{\infty} \left(\frac{1}{2j}\right)^{n} u[n] e^{-j\omega n} + \sum_{n=-\infty}^{-1} \left(\frac{1}{2j}\right)^{-n} u[-n+1] e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{2j} e^{-j\omega}\right)^{n} + \sum_{n=-\infty}^{-1} \left(2j\, e^{-j\omega}\right)^{n}$$

$$\sum_{n=-1}^{-\infty} (\ )^n = \frac{1}{\sum_{n=1}^{\infty} (\ )^n}$$

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$$\sum_{k=0}^{\infty} (2j e^{-j\omega})^{k+1} = \sum_{k=0}^{\infty} (2je^{-j\omega})(2je^{-j\omega})^{k}$$

$$2je^{-j\omega} \sum_{k=0}^{\infty} (2je^{-j\omega})^{k}$$

$$= 2je^{-j\omega}\left(\frac{1}{1-2je^{-j\omega}}\right)$$

substitude back

$$= \frac{1}{1-\frac{1}{2j}e^{-j\omega}} + \frac{1}{\left[2je^{-j\omega}\left(\frac{1}{1-2je^{-j\omega}}\right)\right]}$$

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shifting $u(-t)$ in CT

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$$= \int_{0}^{\infty} e^{t(-1-j\omega)} dt + \int_{-\infty}^{-1} e^{t(-1-j\omega)} dt$$

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↓ property 7 ↓ property 10

$$y(t) = e^{-t}u(t) - e \cdot e^{-(t+1)}u(t+1)$$

-2 because it should be e^{+}
-2 because it was a time-shift

5

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let $x[n]$

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Let $x[n] = 1$

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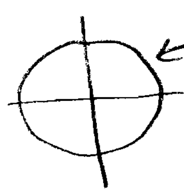

$\leftarrow e^{-jwn}$

e^{-jwn} is a periodic signal, and it has period of 2π.

when you take the Fourier transform of an input $x[n]$, you multiply it by e^{-jwn}, a periodic function, and sum the result over a period of time. That must give you a periodic result/function

10

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$$(j\omega)^2 Y(\omega) + 5j\omega\, Y(\omega) + 6\, Y(\omega) = j\omega\, X(\omega) + 4X(\omega)$$

$\downarrow \mathcal{F}^{-1}$ $\qquad\qquad\qquad\qquad$ $\downarrow \mathcal{F}^{-1}$

& by ~~equation~~ property 16

$$\frac{d^2}{dt^2} y(t) + 5\frac{d}{dt} y(t) + 6y(t) = \frac{d}{dt} x(t) + 4x(t)$$

0

(10 pts) **5.** A CT signal $x(t)$ has Fourier transform

$$\mathcal{X}(\omega) = -2e^{(j-1)\omega}u(\omega+1).$$

$x(t-6)$

Denote by $y(t)$ the signal obtained by delaying $x(t)$ by six seconds. Sketch a graph representing the magnitude $|\mathcal{Y}(\omega)|$ of the Fourier transform $\mathcal{Y}(\omega)$ of $y(t)$. (Justify your answer.)

$$x(t) = \mathcal{F}^{-1}(X(\omega)) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} -2e^{(j-1)\omega}\, u(\omega+1)\, e^{j\omega t}\, d\omega$$

$$= -\frac{1}{\pi}\int_{-1}^{\infty} e^{(j-1)\omega}\, e^{j\omega t}\, d\omega$$

$$= -\frac{1}{\pi}\int_{-1}^{\infty} e^{\omega(j-1+jt)}\, d\omega$$

$$= -\frac{1}{\pi}\left.\frac{e^{\omega(j-1+jt)}}{j-1+jt}\right]_{-1}^{\infty}$$

$$= +\frac{1}{\pi}\,\frac{e^{-(j-1+jt)}}{j-1+jt}$$

$$y(t) = x(t-6) = \frac{1}{\pi}\,\frac{e^{-[j-1+j(t-6)]}}{j-1+j(t-6)}$$

$$= \frac{1}{\pi}\,\frac{e^{-(j-1+jt-6j)}}{j-1+jt-6j}$$

$$= \frac{1}{\pi}\,\frac{e^{-(jt-5j-1)}}{jt-5j-1}$$

$$Y(\omega) = \mathcal{F}(y(t)) = \int_{-\infty}^{\infty} \frac{1}{\pi}\,\frac{e^{-(jt-5j-1)}}{jt-5j-1}\, e^{-j\omega t}\, dt$$

6

$$= \frac{e}{\pi}\int_{-\infty}^{\infty} \frac{e^{-jt}e^{5j}}{jt-5j-1}\, e^{-j\omega t}\, dt$$

$$= \frac{e}{\pi}\int_{-\infty}^{\infty} e^{t(-j-j\omega)}\, \frac{e^{5j}}{jt-5j-1}\, dt$$

$=$