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Recurrence of Inhomogeneous

(*) $a_n = \underbrace{C_{n-1}a_{n-1} + \cdots + C_{n-k}a_{n-k}}_{\text{homogeneous part}} + F(n)$

(1) solve homogenous by finding the roots of characteristic roots

$\lambda_1, \lambda_2, \ldots, \lambda_d$ & multiplicities $e_1, e_2, \ldots, e_d$.

Then

$$a_n = \sum \lambda_i^n \,(\text{generaic polynomial of } n \text{ of degree } e_i - 1)$$

(2) Solve inhomogeneous part

If $F(n) = S^n \cdot$ (a polynomial in n of degree t)

our strategy for finding one particular solution is as follows:

(1) Find the multiplicities m_1 of the factor $(\lambda - s)$ in the characteristic roots.

(2) Write down the following expression:

$$S^n \cdot n^m \cdot (\text{a generic polynomial in } n \text{ of degre } t)$$

(3) Plug into the recursion and find the undetermined coefficients/parameters

Then, the resulting solutions will be a particular soln to the recursion relations.

e.g. $a_n = 2a_{n-1} + 3^n$; $a_0 = 6$

0 → homogeneous

$a_n^h = 2a_{n-1}$; $\chi(\lambda) = (\lambda - 2)$

then, $\lambda = 2$ $e = 1$ and

$a_n^h = 2^n b_0$

* Do not determine b_0 at this point: b_0 will be found in inhomg. algorithm.

1 → specific solution

$F(n) = 3^n$ s = power coef = 3

t = pow of poly = 0 (no polynomial (1))

m = ~~degree~~ multiplicity of s = 0 (3 is not a charac. root)

2 → $a_n^p = 3^n \cdot n^{(0)} \cdot (c_0) = c_0 3^n$

3 → Plugging in...

$a_n = 2a_{n-1} + 3^n = c_0 3^n = c_0 2(3^{n-1}) + 3^n$ / divide by 3^{n-1}

$3c_0 = c_0 2 + 3$

$c_0 = 3$

$$\Rightarrow \quad a_n^p = 3 \cdot 3^n = 3^{n+1}$$

combing the results of homogeneous & particular solution:

$$\underbrace{a_n^p + a_n^h}_{a_n} = \underbrace{(a_{n-1}^p + a_{n-1}^h)}_{a_{n-1}} + 3^n \qquad (\text{* is this necessarily true?})$$

note: a_n^h = capable of fitting init solution, but not quite true ($+3^n$ omitted)

a_n^p = no open parameter but is a true solution.

w/ initial conditions:

$$a_n = b\,2^n + 3^{n+1} \ ; \ \cancel{n=0} \quad a_0 = 6$$

$$6 = b_0 2^0 + 3^1 = b_0 + 3 \ ; \quad \underline{\underline{b_0 = 3}}$$

thus

$$a_n = 3 \cdot 2^n + 3^{n+1}$$

e.g.

$$a_n = a_{n-1} + n \qquad a_0 = 0$$

note:

$$a_n = \sum_{i=0}^{n} i$$

(0) Homogeneous:

$$a_n = a_{n-1} \ : \quad \chi(\lambda) = (\lambda - 1) \ : \quad \lambda = 1 \quad e = 1$$

$$a_n^h = 1^n (b_0) = b_0$$

(1) Inhomogeneous

$$F(n) = n = s^n (\text{polynomial of degree } t)$$

thus: $s = 1$

$m = 1$

$t = 1$

suggested solution: $s^n \cdot n^m \cdot (\text{polynomial of degree } t)$

$$= 1^n \cdot n^{m1} \cdot (c_0 + n c_1)$$

$$= n(c_0 + n c_1)$$

plugging in: $a_n = a_{n-1} + n := c_0 n + c_1 n^2 = c_0 (n-1) + c_1 (n-1)^2 + n$

$$= c_0 n - c_0 + c_1 n^2 - c_1 2n + c_1 + n$$

recombining;

$$0 = n^2 (c_1 - c_1) + n(c_0 - 2c_1 + 1 - c_0) + 1(-c_0 + c_1)$$

we want our final ~~part~~ ↑ to ~~appear as F(n)=n~~ be zeros..

$-2c_1 + 1 = 0 \longrightarrow c_1 = \frac{1}{2}$

$-c_0 + c_1 = 0 \longrightarrow c_0 = \frac{1}{2}$

$\Rightarrow a_n^p = \frac{1}{2}n + \frac{1}{2}n^2$

combing homogeneous & inhomogeneous

$a_n = a_n^p + a_n^h = \frac{1}{2}n^2 + \frac{1}{2}n + b_0$

applying initial condition

$0 = \frac{1}{2}(0^2) + \frac{1}{2}(0) + b_0 \quad ; \quad b_0 = 0$

$\boxed{a_n = \frac{1}{2}n^2 + \frac{1}{2}n}$

(which is what little Gauss figured out ...)

Eg 3

$a_n = 3a_{n-1} - 2a_{n-2} + 3 \qquad a_0 = 0 \qquad a_1 = 1$

0. Homogeneous

$\chi(\lambda) = \lambda^2 - 3\lambda + 2 = (\lambda-2)(\lambda-1)$

$\lambda_1 = 1 \qquad e_1 = 1$

$\lambda_2 = 2 \qquad e_2 = 1$

$a_n^h = 1^n(b_0) + 2^n(b_1) = b_0 + 2^n b_1$

1. Inhomogeneous

$F(n) = 3 = s^n \cdot \text{(polynomial of deg. } t)$

$s = 1 \quad m = 1 \quad t = 0$

suggested solution $= 1^n n (c_0) = c_0 n$

plugging in: $c_0 n = 3c_0(n-1) - 2c_0(n-2) + 3$

$= 3c_0 n - 3c_0 - 2c_0 n + 4c_0 + 3$

$0 = n(3c_0 - 2c_0 - c_0) + 1(4c_0 - 3c_0 + 3)$

$4c_0 - 3c_0 + 3 = 0$

$c_0 = -3$

$a_n^p = -3n$

$a_n = a_n^p + a_n^h = 2^n b_1 + -3n + b_0$

w/init. cond.

$\left.\begin{array}{l} 0 = b_1 + b_0 \\ 1 = 2b_1 - 3 + b_0 \end{array}\right] \Rightarrow \begin{array}{l} b_0 = 4 \\ b_1 = -4 \end{array}$

$\underline{a_n = 4(2^n) - 4 - 3n}$

Consider

$F(n) = n + 2^n$ ... what is s? (we need in form s^n(poly))

$F_1(n) = n$ $\quad s=1 \quad t=1 \quad m=1 \quad a_n^{p_1} = n(c_0 + c_1 n)$

$F_2(n) = 2^n$ $\quad s=2 \quad t=0 \quad m=0 \quad a_n^{p_2} = 2^n c_2$

$$\boxed{a_n^p = n(c_0 + c_1 n) + 2^n c_2}$$

Generating Function