Fill-in-the-blanks notes for
ECE301
Fall 2018
Prof. Boutin

	Part I- Signals, Systems, and Convolution	
		O.W.N. References
1	Definitions of CT/DT signals and systems	1.0, 1.1.1. 1.5.1
2	Signal power and energy	1.1.2
3	Basic systems: transformation of the independent variable	1.2.1, 1.2.3
4	Periodic signals	1.2.2
5	Basic signals: exponential, sine, unit impulse, unit step	1.3,1.4
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7	DT and CT convolution	2.1,2.2

Test 1- covers Part I

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Test 3-covers Part III except ___________________

Final Exam- Covers Part I, II, III

1. CT and DT signals and systems: definitions and examples

Recall signal (= function)

CT = "continuous time"

$$\underset{\text{real number}}{t} \longmapsto x(t)$$

t
↓
continuously varying variable
$t \in \mathbb{R}$

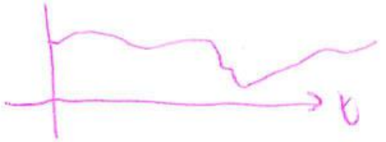
DT = "discrete time"

$$\underset{\text{integer}}{n} \longmapsto x[n]$$

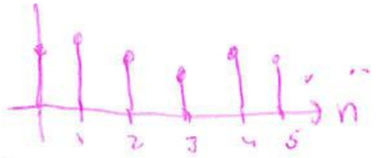
n
↓
discrete variable
ints
$n = \dots -3, -2, -1, 0, 1, 2 \dots$
$n \in \mathbb{Z}$

Examples

$x(t)$ — ... temp at time t, where $t = 0$ noon, Jan 1, 1900 @ Purdue Airport



$x_d[n]$ = temp at day n, $n = 0$ Jan 1, 1900 at noon



MATLAB sound signals examples

Pre-recorded music:

```
>> load handel
>> sound (y, Fs)
>> plot(y)
>> figure (2)
>> plot ( y (1:5) )
```

"A" 440 hz :

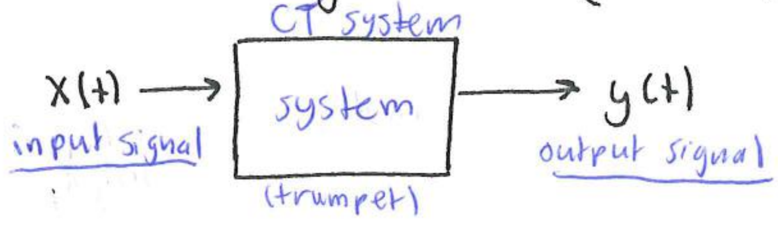
```
>> clear
>> delta = 0.00005 ;
>> t = 0 : delta : 3 ;
>> f = sin ( 2 * pi * 440 * t ) ;
>> sound (f, 1/delta)
>> plot (f)
>> figure (2)
>> plot( f (1: 5) )
```

~~d~~ delta = $\frac{1}{8192}$

8-22

Systems transform signals into (modified) signals

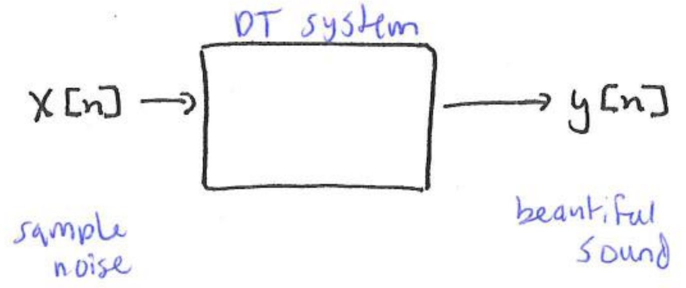
CT



headphones → analog
CT

Signal
↳ takes real #'s and makes them complex

DT



Discrete signals
↳ approximation of CT signals

Examples

→ trumpet

$y[n] = 100\,x[n]$ ↑ loud

$y(t) = \frac{x(t)}{100}$ ↑ less loud

Q?: HW

lim from L and R exist and they are the same

CT signal can be discontinuous as a function
↳ independent variable

Continuous signal → dependent variable varies continuously

CT → calc (easier)

★ Will be on test

2. Signal Power and Energy

	CT	DT → (+1)
Definition: **Energy** expanded by a signal over a time interval: t_1 to t_2	$\int_{t_1}^{t_2} \lvert x(t)\rvert^2 dt$ complex norm $z=\sqrt{Re(z)^2+Im(z)^2} = \sqrt{z \cdot \bar{z}}$ *real non-neg $t_1 < t_2$	$\sum_{n=n_1}^{n_2} \lvert x[n]\rvert^2$ $n_1 < n_2$
Definition: **Average power** of a signal over a time interval: * energy ÷ length of integral	$\left(\frac{1}{t_2-t_1}\right)\int_{t_1}^{t_2} \lvert x(t)\rvert^2 dt$ $t_1 < t_2$	$\frac{1}{(n_2-n_1+1)}\sum_{n=n_1}^{n_2} \lvert x[n]\rvert^2$ $n_1 < n_2$
Definition: **Total energy** E_∞ of a signal	$\int_{-\infty}^{\infty} \lvert x(t)\rvert^2 dt$	$\sum_{n=-\infty}^{\infty} \lvert x[n]\rvert^2$
Definition: **Signal Power** P_∞	$\lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \lvert x(t)\rvert^2 dt$	$\lim_{N\to\infty} \frac{1}{2N+1}\sum_{n=-N}^{N} \lvert x[n]\rvert^2$

★ MUST BE:
real &
non-negative

BAD! NO! $\longrightarrow \frac{1}{\infty}\int_{-\infty}^{\infty} |x(t)|^2 dt \rightarrow \frac{1}{\infty}\sum_{n=\infty}^{\infty} |x[n]|^2$

Ex.1.) $x[n] = j^n$ ***DT**

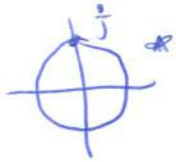
$$E_\infty = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

$$= \sum_{n=\infty}^{\infty} x^*[n] \cdot x[n]$$ ← conjugate

$$= \sum_{n=-\infty}^{\infty} \underbrace{(-j)^n j^n}$$

$$= \sum_{n=-\infty}^{\infty} (-j^2)^n$$

$$E_\infty = \sum_{n=-\infty}^{\infty} 1^n = \sum_{n=-\infty}^{\infty} 1 = \boxed{\infty}$$



* complex plane

~~[illegible]~~

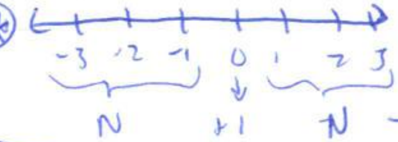
$$P_\infty = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \underbrace{|x[n]|^2}_{1}$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} 1$$ ⊛

$$= \lim_{N\to\infty} \frac{1}{\cancel{2N+1}} \cancel{2N+1}$$

$$\boxed{P_\infty = 1}$$ → has to be positive real number!

* infinite energy, finite power

⊛


→ 2N+1

Ex.2) ***CT**

$$x(t) = \begin{cases} -2, & 0 \le t \le 5 \\ 0, & \text{else} \end{cases}$$

$$E_\infty = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_0^{5} |-2|^2 dt$$

$$= \int_0^5 4\,dt = \boxed{20 = E_\infty}$$

* Finite energy
↳ ZERO power

$$P_\infty = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt$$

$$= \lim_{T\to\infty} \frac{1}{2T} \int_0^5 |-2|^2 dt + \overbrace{\int_{-\infty}^{0} |0|^2 dt + \int_5^{\infty} |0|^2 dt}^{0}$$

* split integral into intervals where its known → its 0 here so ignore

$$= \lim_{T\to\infty} \frac{1}{2T} \int_0^5 4\,dt = \lim_{T\to\infty} \frac{1}{2T} \cdot 20$$

$$= \lim_{T\to\infty} \frac{10}{T} = 0$$

Back to ex 1

$$P_\infty = \lim_{N\to\infty} \frac{1}{2N+1}$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} \cdot \underbrace{\cancel{\sum_{n=-\infty}^{\infty} 1}}_{\infty}$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} \cancel{\infty}$$

$$0 \cdot \cancel{\infty} = 1?$$

common mistake

* Always compute ENERGY first

Ex3 →

real # → no "j"
↑
$E_\infty \geq 0$ *norm

Check: • $E_\infty, P_\infty \geqslant 0$ since → $|x(t)|^2 \geq 0$, for all t

$$\Rightarrow \int_{t_1}^{t_2} |x(t)|^2 \geq 0$$

• If $P_\infty > 0$, then $E_\infty = \infty$

FACT * → if E_∞ is finite, then $P_\infty = 0$

assum E_∞ is finite

then $P_\infty = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 \, dt$

$= \lim_{T \to \infty} \frac{1}{2T} \cdot \lim_{T \to \infty} \int_{-T}^{T} |x(t)|^2 \, dt$

↑ since both limits exist and are finite

$= 0 \cdot E_\infty$

$P_\infty = 0$

WARNING: Never split into two factors

$$P_\infty = \left(\lim_{N \to \infty} \frac{1}{2N+1} \right) \left(\lim_{N \to \infty} \sum_{n=-N}^{N} |x[n]|^2 \right)$$

unless both factors are finite !

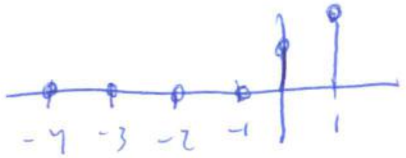
i.e. unless energy E_∞ is finite

otherwise you will get stuck with

$P_\infty = 0 \cdot \infty = ?$

Ex 3)

$$x[n] = \begin{cases} 2^{-n}, & n \geq 0 \\ 0, & \text{else} \end{cases}$$



$$E_\infty = \sum_{n=-\infty}^{\infty} |x[n]|^2$$

$$= \underbrace{\sum_{n=\cancel{-\infty}\,0}^{\infty} |2^{-n}|^2}_{\text{= for only (+) values of n}} + \cancel{\sum_{n=-\infty}^{-1} 0} \text{ zero}$$

$$= \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n = \frac{1}{1-\frac{1}{4}} = \frac{4}{3}$$

infinite
✱ geo. series

$$\boxed{E_\infty = \frac{4}{3}}$$

✱ finite so power = 0

$$P_\infty = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x[n]|^2$$

$$= \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=0}^{N} \left(\frac{1}{4}\right)^n$$

$$= \lim_{N\to\infty} \underbrace{\frac{1}{2N+1}}_{0} \cdot \frac{1 - \left(\frac{1}{4}\right)^{N+1} \to 0}{1-\frac{1}{4}}$$

$$\boxed{P_\infty = 0}$$

✱ infinite duration

$$\sum_{n=0}^{N} r^n = \underbrace{1 + r + r^2 + r^3 + \dots r^n}_{S}$$

✱ Geometric series

$$(1-r)\cdot S = 1\cdot S - r\cdot S$$

$$= 1 + \cancel{r} + \cancel{r^2} + \cancel{r^3} + \dots \cancel{r^N}$$

$$- \cancel{r} - \cancel{r^2} - \cancel{r^3} - \dots \cancel{r^N} - r^{N+1}$$

$$= 1 - r^{N+1}$$

if $r \neq 1$

$$S = \frac{1 - r^{N+1}}{1-r} \xrightarrow{N\to\infty} \begin{cases} \frac{1}{1-r}, & \text{if } |r|<1 \\ \text{diverges}, & \text{else} \end{cases}$$

if $r = 1$

$$S = N+1 \xrightarrow{N\to\infty} \infty \text{ (diverges)}$$

$$\text{so } \sum_{n=0}^{\infty} r^n = \begin{cases} \frac{1}{1-r}, & \text{if } |r|<1 \\ \text{diverges}, & \text{else} \end{cases}$$

$\sum_{n=0}^{\infty} \left(\frac{1}{10}\right)^n$ $r = 1/10$, $|r| < 1$

$\sum_{n=0}^{\infty} 2^n$ $r = 2$, $|r| > 1$

$\sum_{n=0}^{\infty} (-2)^n$ $r = -2$, $|r| > 1$

$\sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n$ $r = -1/2$, $|r| < 1$

3. Basic Systems: transformations of independent variable

CT | DT

Time delay by t_0 $(t_0 \in \mathbb{R})$

$x(t) \longrightarrow \boxed{} \longrightarrow y(t) = x(t-t_0)$ $x[n] \longrightarrow \boxed{} \longrightarrow y[n] = x[n-n_0]$

$(+)$ $t_0 \rightarrow$ right
$(-)$ $t_0 \rightarrow$ left

Time reversal

$x(t) \longrightarrow \boxed{} \longrightarrow y(t) = x(-t)$ $x[n] \longrightarrow \boxed{} \longrightarrow y[n] = x[-n]$

Time scaling

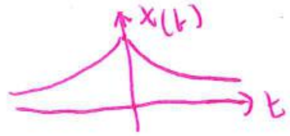
$x(t) \longrightarrow \boxed{} \longrightarrow y(t) = x(at)$ $x[n] \longrightarrow \boxed{} \longrightarrow y[n] = x[an]$

$a \in \mathbb{R}_{\geq 0}$

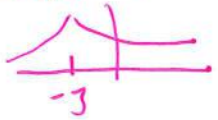
$a \neq 0$
a realnumber

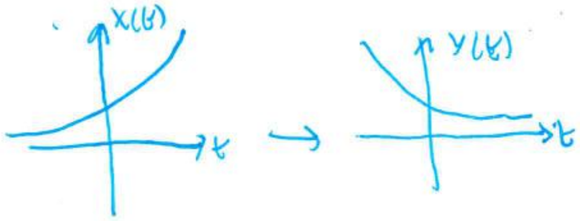
a integer
$a > 0$

Illustrations

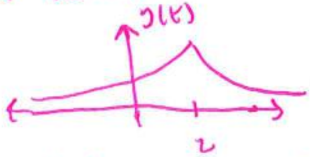


if $t_0 = -3$

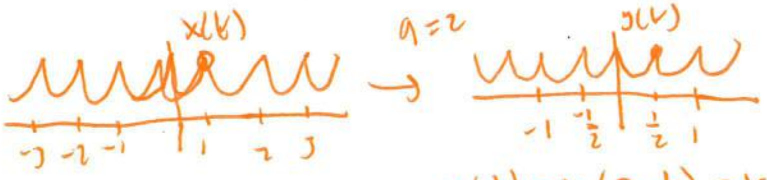




if $t_0 = 2$

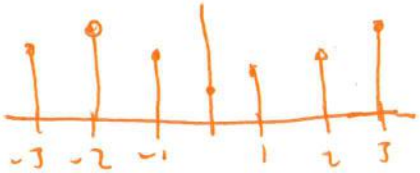


$y(2) = x(2-2) = x(0)$



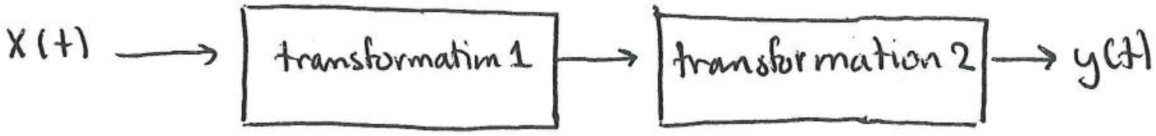
$y(\frac{1}{2}) = x(2 \cdot \frac{1}{2}) = x(1)$

* tricky in DT bc has to be integers



$\rightarrow$ $a = \frac{1}{3}$
$y[1] = x[\frac{1}{3} \cdot 1] = x[\frac{1}{3}]$
$\rightarrow$ can't do it! missing info → not known

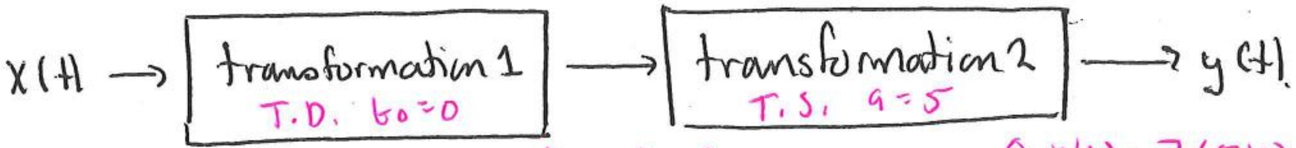
We consider cascades of transformations of independent variable.



Example.

transformation 1: $y(t) = x(t-2)$ → time delay

transformation 2: $y(t) = x(5t)$ → time scaling

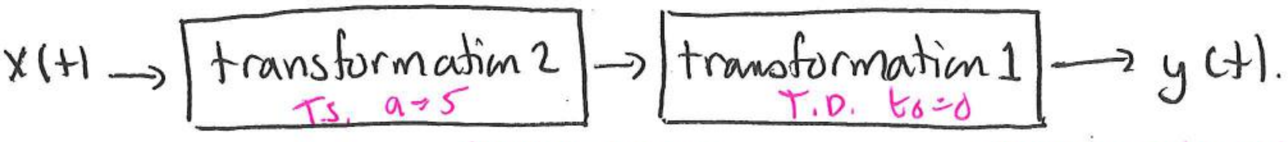


① $z(t) = x(t-2)$

② $y(t) = z(5t)$

③ $z(\square) = x(\square-2)$

$\rightarrow y(t) = x(5t-2)$



① $z(t) = x(5t)$

② $y(t) = z(t-2)$

③ $z(\square) = x(5\square)$

$\rightarrow y(t) = x(5\cdot(t-2))$

$y(t) = x(5t-10)$

✱ order is important!
- very tricky
- "not what you'd think"

even → unchanged
odd → reversed

Even / odd Signals

	CT	DT
Definition: we say a signal is **even** if it is unchanged under a time reversal.	$x(t) = x(-t)$	$x[n] = x[-n]$
Definition: we say a signal is **odd** if its sign is merely reversed under a time reversal.	$-x(t) = x(-t)$	$-x[n] = x[-n]$.

Lemma Any (C.T. or D.T.) signal can be written as a sum of an even signal and an odd signal.

proof $$x(t) = \underbrace{\frac{x(t) + \overset{\text{added}}{x(-t)}}{2}}_{x_e(t)} + \underbrace{\frac{x(t) - x(-t)}{2}}_{x_o(t)}$$

"even part" "odd part"

check that

$$x_e(-t) = \frac{x(-t) + x(t)}{2} = \frac{x(t) + x(-t)}{2} = x_e(t), \therefore \text{even} ✓$$

$$x_o(-t) = \frac{x(-t) - x(t)}{2} = -\left(\frac{x(t) - x(-t)}{2}\right) = -x_o(t), \therefore \text{odd} ✓$$

- time reverse
- add (or sub)
- div by 2

ex) $x(t) = 0$ for all t
↳ is both even + odd

How to tell that a signal is even? Check that odd part is zero.

How to tell that a signal is odd? Check that even part is zero.

Part I

4. Periodic Signals

	CT	DT
Definition: We say a signal is <u>periodic</u> if	there exists $T > 0$ s.t. $x(t+T) = x(t)$ for all t.	there exists $N > 0$ s.t. $x[n+N] = x[n]$ for all n.
	T = "period" not unique	N = "period" * period has to be an integer

> The "fundamental period" of a signal is the smallest among all periods of the signal.

Examples:

$x(t) = \cos(t)$ period 2π

bc $x(t+2\pi) = \cos(t+2\pi)$
$= \cos(t)$
$= x(t)$

$x[n] = j^n$ period 4

bc $x[n+4] = j^{n+4}$
$= j^n \cdot j^4$
$= j^n$
$= x[n]$

* check n=1-j | 5-j
2-1 | 6-1
3-j | 7-j
4-1 | 8-1

Observation

$j = e^{j\frac{\pi}{2}}$

$j^n = e^{j\frac{\pi}{2}n} = e^{j\frac{\pi}{4}n}$

$x[n] = e^{j\frac{2\pi}{4}n}$ ← period 4

likewise

$x[n] = e^{j\frac{2\pi}{5}n}$ ← period 5

Important!

Periodic "repetition" of a signal.

Question: Is the signal $x(t) = \sum_{k=-\infty}^{\infty} e^{-(t+5k)^2}$ periodic?

Observe that

$$x(\underline{t+5}) = \sum_{k=-\infty}^{\infty} e^{-(\underline{t+5}+5k)^2}$$

$$= \sum_{k=-\infty}^{\infty} e^{-(t+5(k+1))^2} \quad \nwarrow \text{, let } n=k+1$$

$$= \sum_{n=-\infty}^{\infty} e^{-(t+5n)^2} \quad \text{, let } k=n$$

$$= \sum_{k=-\infty}^{\infty} e^{-(t+5k)^2}$$

$$= x(t) \text{, for all } t$$

so → yes. Period 5

shifts 5 and add

In general, if g(t) is a signal then $x(t) = \sum_{k=-\infty}^{\infty} g(t+Tk)$ $= rep_T(g(t))$ is periodic with period T.

5. Important Signals: exponential, sine, unit impulse, unit step.

	CT	DT
General form of a complex exponential signal	$x(t) = Ce^{at}$	$x[n] = Ca^n$

C, a complex numbers

Examples : $x(t) = \overset{\text{magnitude } 1}{\overset{\downarrow}{1}} e^{jt} \leftarrow \text{phase } t$

$x(t) = (1+3j)e^{jt}$

$x(t) = e^{(1+5j)t} = \underset{\text{magnitude}}{\underline{\underline{e^t}}} e^{5jt}$

$x(t) = \underline{e^t}$ mag, phase 0

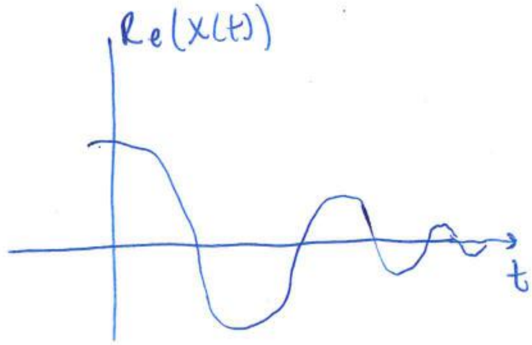
e^{jt} periodic w/ period 2π

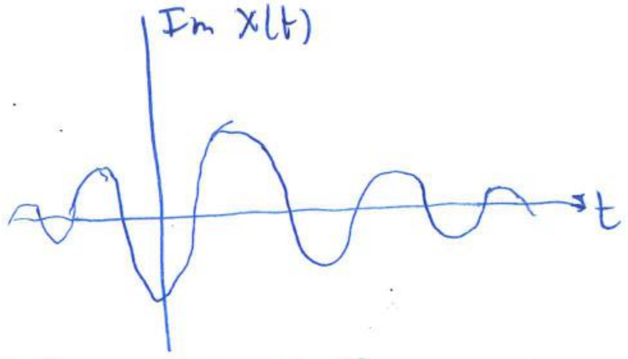
* $e^{j\omega_0 t} \longrightarrow \frac{2\pi}{|\omega_0|}, \ \forall \omega_0 \in \mathbb{R}$

Not all complex exponentials are periodic

ex) $x(t) = e^{(-1+j)t} = e^{-t} e^{jt}$

$= e^{t}(\cos t + j \sin t)$





* damped oscillation
dissipates energy

Recall: B'Euler up!!!

$$\boxed{e^{j\theta} = \cos\theta + j\sin\theta}, \quad \theta \in \mathbb{R}$$

$$\Rightarrow \cos\theta = \mathrm{Re}(e^{j\theta}) = \frac{e^{j\theta} + e^{-j\theta}}{2} \qquad \sin\theta = \mathrm{Im}(e^{j\theta}) = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

must know ✱

check $e^{j\theta} + e^{-j\theta} = \cos\theta + j\sin\theta + \cos(-\theta) + j\sin(-\theta)$

$= \cos\theta + \cancel{j\sin\theta} + \cos\theta + -\cancel{j\sin\theta}$

$= 2\cos\theta$

similarily ... $e^{j\theta} - e^{-j\theta} = \dots$

$= 2j\sin\theta$

Recall: A complex number z can be written in polar coordinates $z = |z|e^{j\theta}$.

$$\Rightarrow \quad x[n] = Ca^n = |c|\,|a|^n e^{j(\omega n + \varphi)}$$

initial phase at n=0

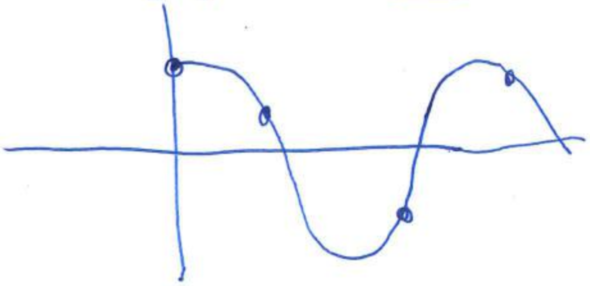
controls freq. of oscillations

controls growth/decay of envelope of real/imag parts

where $C = |c|e^{j\varphi}$

$a = |a|e^{j\omega}$

✱ Oscillating part is <u>NOT</u> necessarily periodic because of sampling pattern



↗ n DT

When is $x[n] = e^{j\omega n}$ periodic?

check $x[n+N] = x[n]$, $\forall n$

$$\iff e^{j\omega(n+N)} = e^{j\omega n}$$

$$\iff \cancel{e^{j\omega n}} \cdot e^{j\omega N} = \cancel{e^{j\omega n}}$$

$$\iff e^{j\omega N} = 1$$

$$\iff \omega N \rightarrow \text{is a multiple of } 2\pi$$

$$\iff \omega N = k\, 2\pi \rightarrow \text{for some integer } k \in \mathbb{Z}$$

$$\left[\frac{\omega}{2\pi} = \frac{k}{N}\right], \rightarrow \text{for some integer } k \in \mathbb{Z}$$

$$\boxed{e^{j\omega n} \text{ is periodic if and only if } \frac{\omega}{2\pi} \text{ is a rational number}}$$

Examples: e^{jn} not periodic

bc $\omega = 1 \rightarrow \frac{\omega}{2\pi} = \frac{1}{2\pi}$ * not rational

note: $e^{jn} = \cos n + j \sin n$ ← not periodic (cos n), not periodic (sin n)

$e^{\frac{1}{2}j\pi n}$ is periodic

bc $\omega = \frac{1}{2}\pi \rightarrow \frac{\omega}{2\pi} = \frac{1}{4}$ * is rational

Part I

What is the fundamental period N_0 of the DT signal $x[n] = e^{j\omega n}$?

(assuming $\frac{\omega}{2\pi}$ is rational)

It is the smallest positive integer N_0 such that

$$x[n+N_0] = x[n] \text{ for all } n.$$

$$\Longleftrightarrow \quad \omega N_0 = K\,2\pi \quad , K \in \mathbb{Z}$$

$$\Longleftrightarrow \quad N_0 = K\frac{2\pi}{\omega} \quad , K \in \mathbb{Z}$$

The fundamental period of $x[n] = e^{j\omega n}$ is $N_0 = K \; \frac{2\pi}{\omega}$ where K is the smallest positive integer that makes $K\frac{2\pi}{\omega}$ an integer

Observe: If $x_1[n]$ has fundamental period N_1
$x_2[n]$ N_2

then $\underline{x_1[n] + x_2[n]}$ is periodic with period

$\longrightarrow N = LCM(N_1, N_2)$

but this may <u>not</u> be the <u>fundamental period.</u>

Part I

$x(t) = e^{j\frac{2\pi}{T}t}$ → period T | $x[n] = e^{j\frac{2\pi}{N}n}$ → period N

Harmonically Related Exponentials

CT *period T

$$\left\{ x_k(t) = e^{jk\frac{2\pi t}{\cancel{N}T}} \right\}_{k \in \mathbb{Z}}$$

$$e^{jk(\frac{2\pi}{T}(t+T))} = e^{jk(\frac{2\pi t}{T}) + jk2\pi}$$

$$= e^{jk\frac{2\pi t}{T}} \cdot 1$$

$$= e^{jk\frac{2\pi}{T}t}$$

DT *period N

$$\left\{ x_k[n] = e^{jk\frac{2\pi}{N}n} \right\}_{k \in \mathbb{Z}}$$

*finite n

$$e^{jk\frac{2\pi}{N}(n+N)} = e^{jk\frac{2\pi}{N} + jkn}$$

$$= e^{jk\frac{2\pi}{N}} \cdot 1$$

$k = 1$ $x_1(t)$ period T

$k = 2$ $x_2(t)$ period $\frac{T}{2}$ → $e^{j2\frac{2\pi}{T}(t + \frac{T}{2})} = e^{j2\frac{2\pi}{T}t} \underbrace{e^{j2\pi T}}_{1}$

$= e^{j2\frac{2\pi}{T}t}$

⋮

k $x_k(t)$ period $\frac{T}{k}$

Matlab → adding sine w/ harmonic frequencies + changing coefficents to create different sounds

```
delta = 1/8192
t=0:delta:2;
x = sin(2*pi*256*t)
sound(x, 8192)
x= sin(2*pi*256*t) + 20 sin(2*pi*256*2*t)
                   + 30 sin(2*pi*256*3*t)
                   + 100 sin(2*pi*256*4*t);
```

→ change coefficents * changes timber

Part I

Observe: There is a finite number of distinct signals in the set $\left\{ e^{jk\frac{2\pi}{N}n} \right\}_{k \in \mathbb{Z}}$.

because let $x_k[n] = e^{jk\frac{2\pi}{N}n}$

$$\text{then } x_{k+N}[n] = e^{j(k+N)\frac{2\pi}{N}n}$$

$$= e^{jk\frac{2\pi}{N}n} e^{jN\frac{2\pi}{N}n}$$

$$= e^{jk\frac{2\pi}{N}n} \cdot 1$$

$$= x_k[n]$$

So the distinct signals in the set are

$$x_0[n],\ x_1[n],\ x_2[n],\ \cdots,\ x_{N-1}[n].$$

e.g. $N=4$ (period 4 signals)

$x_0[n] = e^0 = 1$

$x_1(n] = e^{j\frac{2\pi}{4}n} = e^{j\frac{\pi}{2}n} = j^n$

$x_2(n] = e^{j2\frac{2\pi}{4}n} = e^{j\pi n} = (-1)^n$

$x_3(n] = e^{j3\frac{2\pi}{4}n} = e^{j\frac{3}{2}\pi n} = (-j)^n$

$x_4(n] = e^{j4\frac{2\pi}{4}n} = e^{j2\pi n} = 1^n = 1$

$x_5[n] = e^{j5\frac{2\pi}{4}n} = e^{j(4+1)\frac{2\pi}{4}n} = e^{j2\pi n} \cdot e^{j\frac{2\pi}{4}n} = e^{j\frac{\pi}{2}n} = j^n$

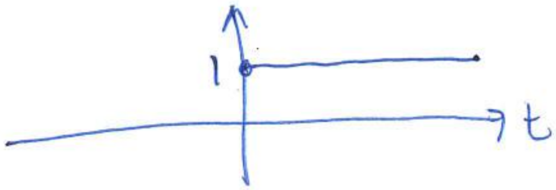
etc...

periodic repitition of <u>signals</u> in a sequence

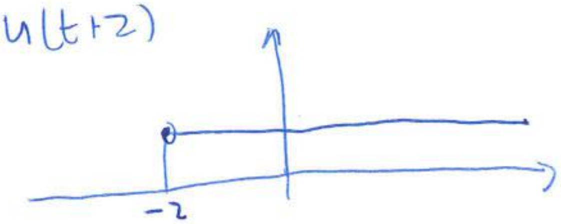
Unit Step Signal

CT

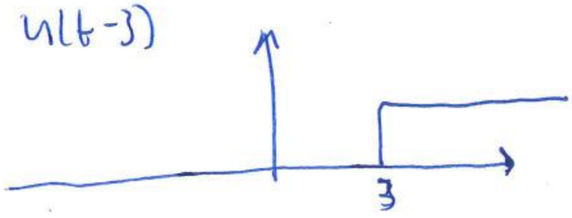
$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$$



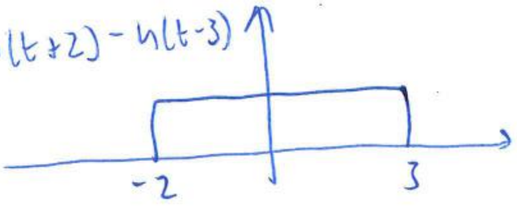
u(t+2)



u(t-3)

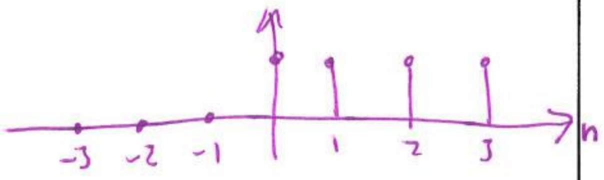


u(t+2) - u(t-3)

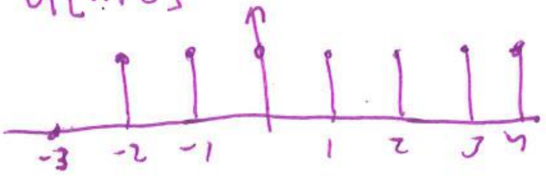


DT

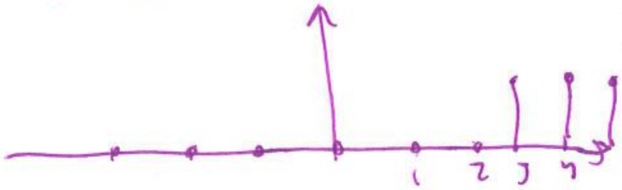
$$u[n] = \begin{cases} 0, & n < 0 \\ 1, & n \geq 0 \end{cases}$$



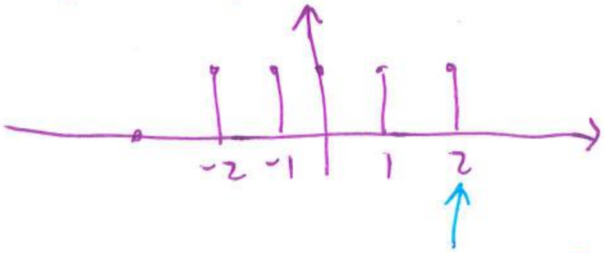
u[n+2]



u[n-3]



u[n+2] - u[n-3]



Stops at 2
(instead of 3)

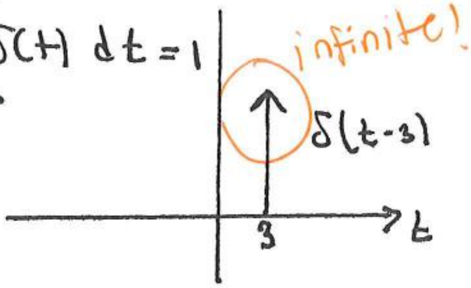
Unit Impulse Signal

$\int_{-\infty}^{\infty} \delta(t)\, dt = 1$

CT

$$\delta(t) = \begin{cases} 0, & t \neq 0 \\ \infty, & t = 0 \end{cases}$$

$\int_{-\infty}^{\infty} \delta(t)\, dt = 1$



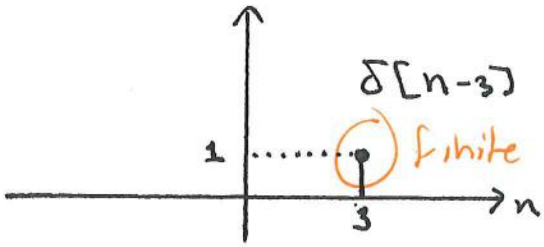
can be better defined as $\lim_{\sigma \to 0} \frac{1}{\sqrt{2\pi}\sigma} e^{-t^2/\sigma^2}$

i.e.) Gassian w/ infinetisimaly smaller σ

Distribution → not function

DT

$$\delta[n] = \begin{cases} 0, & n \neq 0 \\ 1, & n = 0 \end{cases}$$



Relationship Betwen δ and u:

$u(t) = \int_{-\infty}^{t} \delta(t')\, dt'$ let $\tau = t - t'$

or $u(t) = \int_{0}^{\infty} \delta(t-\tau)\, d\tau$ "sum" of shifted deltas

$\delta(t) =$ " $\frac{d}{dt} u(t)$ "

$\{u_n\} \xrightarrow[n \to \infty]{} u(t)$

$\delta(t) = \lim_{n \to \infty} \frac{d}{dt} u_n(t)$

$u[n] = \sum_{k=-\infty}^{n} \delta[k]$ let $k' = n - k$

or $u[n] = \sum_{k=0}^{\infty} \delta[n-k]$ sum of shifted delta signals

$\delta[n] = u[n] - u[n-1]$

6. System Properties

a) Memoryless Systems
b) Invertible Systems
c) Causal Systems
d) Stable Systems
⁎e) Linear Systems
⁎f) Time-invariant Systems

a) Memoryless Systems - Systems with Memory

Definition: A system is called "memoryless" if the output signal at any given time only depends on the input signal at that specific time (not on past or future of input signal).

alt. def: System is memoryless $\Leftrightarrow$ for any $t_0 \in \mathbb{R}$, the output $y(t_0)$ depends only on $x(t_0)$

alt def: System is memoryless $\Leftrightarrow$ If $x(t)$ and $\tilde{x}(t)$ are 2 inputs S.T. $x(t_0) = \tilde{x}(t_0)$, then $y(t_0) = \tilde{y}(t_0)$

$y(t) = 10\,x(t)$ → memoryless
$y(t) = x(t-1)$ → has memory
$y(t) = (t-1)\,x(t)$ → memoryless

$y(t) = f(t, x(t))$
general form for a memoryless system

b) Invertible Systems - Non-invertible Systems

Definition: A system is called "invertible" if distinct input signals yield distinct output signals.

* need a 1 to 1 property

Alt. Defn:

System is invertible $\Longleftrightarrow$ there exists an inverse system such that the cascade

$$x(t) \longrightarrow \boxed{\text{system}} \longrightarrow \boxed{\text{inverse system}} \longrightarrow x(t)$$

* leaves the input signal unchanged

Ex1 $y(t) = 2x(t) + 3$

① isolate $x(t) = \frac{y(t)-3}{2}$

② switch $x \leftrightarrow y$

$$\left. y(t) = \frac{x(t)-3}{2} \right] \text{inverse}$$

③ check

$$x(t) \longrightarrow \boxed{\text{System}} \longrightarrow y(t) = 2x(t)+3 \longrightarrow \boxed{\text{inverse system}} \longrightarrow z(t) = \frac{y(t)-3}{2} = \frac{2x(t)+3}{2} = x(t)$$

Ex2

~~x~~$y(t) = x(t-3)$ can't isolate $x(t)$

$y(t) = x(t+3)$

c) Causal Systems - Non-causal Systems

Definition: A system is called "causal" if the output signal at any given time only depends on the input signal at that time or at previous times (i.e. past and present, not future).

= "non-anticipative" system

Alt. Defn: A system is "causal" if for any t_0, the output $y(t_0)$ only depends on $x(t)$ for $t \leq t_0$

* past OK * future not ok

* All memoryless systems are causal

$y(t) = x(t+1)$ → not causal (- depends on future value

$y(0) = x(1)$

$y(t) = x(t-1)$ → is causal

$y(t) = x(10t)$

↳ $y(1) = x(10)$ → not casusal

$y(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$ → causal, w/ memory

$y(t) = \int_{t}^{\infty} x(\tau)\, d\tau$ → non causal, w/ memory

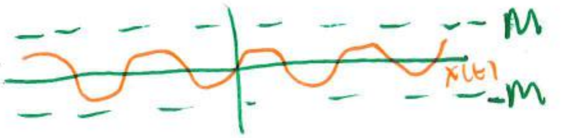
$y(t) = x(t+10)$

$t = 0$

$y(0) = x(10)$ ← future → so NOT causal

d) Stable Systems - Unstable Systems

Definition: A system is called (BIBO) "stable" if bounded inputs yield bounded outputs.



→ $x(t)$ signal is bounded if there exists $M > 0$ such that $|x(t)| < M$

$x(t) = t$ → unbounded

$x(t) = \cos(t) + 2$ → bounded

→ bounded system DNE!
its a system (no such thing)
of bounded signals

$y(t) = e^{x(t)}$ → stable

bc if $|x(t)| < \varepsilon$ (i.e. $x(t)$ bounded)

then $|y(t)| = |e^{x(t)}| < e^{\varepsilon}$ (i.e. $y(t)$ bounded)

$y(t) = t\,x(t)$

take $x(t) = 7$ (bounded input)

$y(t) = t \cdot 7$ (not bounded) 

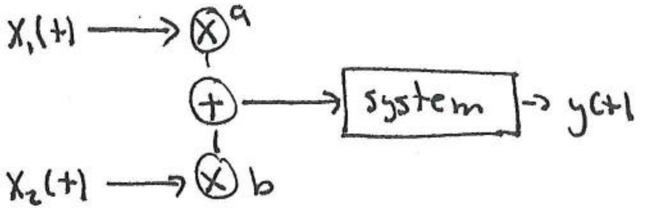
e) Linear Systems - Non-linear Systems

Definition #1 : A system is called "linear" if it commutes with linear combinations.

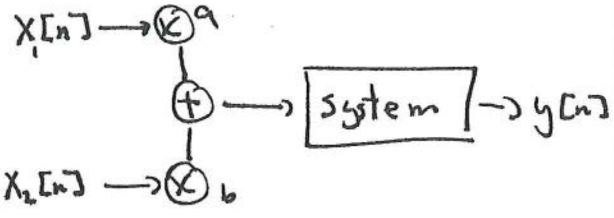
$$a x_1(t) + b x_2(t) \rightarrow \boxed{} \rightarrow a y_1(t) + b y_2(t)$$

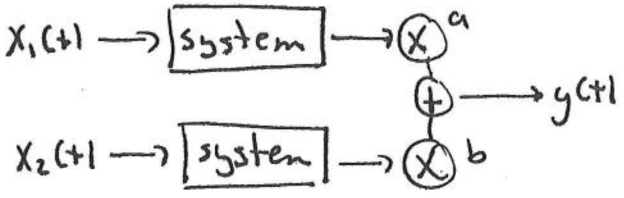
Definition #2 : A system is called "linear" if the following cascades yield the same output signal, for any value of $a, b \in \mathbb{C}$.

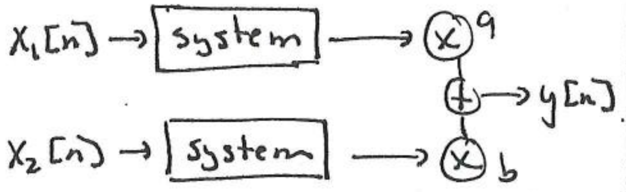
CT



DT







Definition #3 : A system is called "linear" if

CT

$$a x_1(t) + b x_2(t) \rightarrow \boxed{\text{system}} \rightarrow a y_1(t) + b y_2(t)$$

for any $a, b \in \mathbb{C}$

DT

$$a x_1[n] + b x_2[n] \rightarrow \boxed{\text{system}} \rightarrow a y_1[n] + b y_2[n]$$

for any $a, b \in \mathbb{C}$

Definition #4 : A system is called "linear" if for any constants $a, b \in \mathbb{C}$ and for any input signals $x_1(t)$, $x_2(t)$ $(x_1[n], x_2[n])$ yielding output $y_1(t)$, $y_2(t)$ $(y_1[n], y_2[n])$ respectively, the system's response to $a x_1(t) + b x_2(t)$ $(a x_1[n] + b x_2[n])$ is $a y_1(t) + b y_2(t)$ $(a y_1[n] + b y_2[n])$.

Example 1: The system defined by $y[n] = x[-n]$ is linear.

because if

$$X_1[n] \rightarrow \boxed{\text{system}} \rightarrow y_1[n] = X_1[-n]$$

$$X_2[n] \rightarrow \boxed{\text{system}} \rightarrow y_2[n] = X_2[-n]$$

then

* replace arg w "-n"

$$Z[n] = a X_1[n] + b X_2[n] \rightarrow \boxed{\text{system}} \rightarrow Z[-n] =$$

$$= a X_1[-n] + b X_2[n]$$

$$= a y_1[n] + b y_2[n]$$

→ system is linear

✓

Example 2: The system defined by $y[n] = X[n]^2$ is not linear

because if

$$X_1[n] \rightarrow \boxed{\text{system}} \rightarrow y_1[n] = X_1^2[n]$$

$$X_2[n] \rightarrow \boxed{\text{system}} \rightarrow y_2[n] = X_2^2[n]$$

then

$$Z[n] = a X_1[n] + b X_2[n] \rightarrow \boxed{\text{system}} \rightarrow Z^2[n] =$$

$$= (a X_1[n] + b X_2[n])^2$$

$$\neq a X_1^2[n] + b X_2^2[n]$$

$$= a y_1[n] + b y_2[n].$$

→ system not linear

f) Time-invariant Systems - Time-variant Systems

Definition #1 : A system is called "time-invariant" if it commutes with time delays.

Definition #2 : A system is called "time-invariant" if the following cascades yield the same output signal for any value of t_o | n_o

CT

$$x(t) \rightarrow \boxed{\text{system}} \rightarrow \boxed{\text{time shift by } t_0} \rightarrow y(t)$$

$$x(t) \rightarrow \boxed{\text{time shift by } t_0} \rightarrow \boxed{\text{system}} \rightarrow y(t).$$

DT

$$x[n] \rightarrow \boxed{\text{system}} \rightarrow \boxed{\text{time shift by } n_0} \rightarrow y[n]$$

$$x[n] \rightarrow \boxed{\text{time shift by } n_0} \rightarrow \boxed{\text{system}} \rightarrow y[n]$$

Definition #3 : A system is called "time-invariant" if

CT

$$x(t-t_0) \rightarrow \boxed{\text{system}} \rightarrow y(t-t_0)$$

for any $t_0 \in \mathbb{R}$

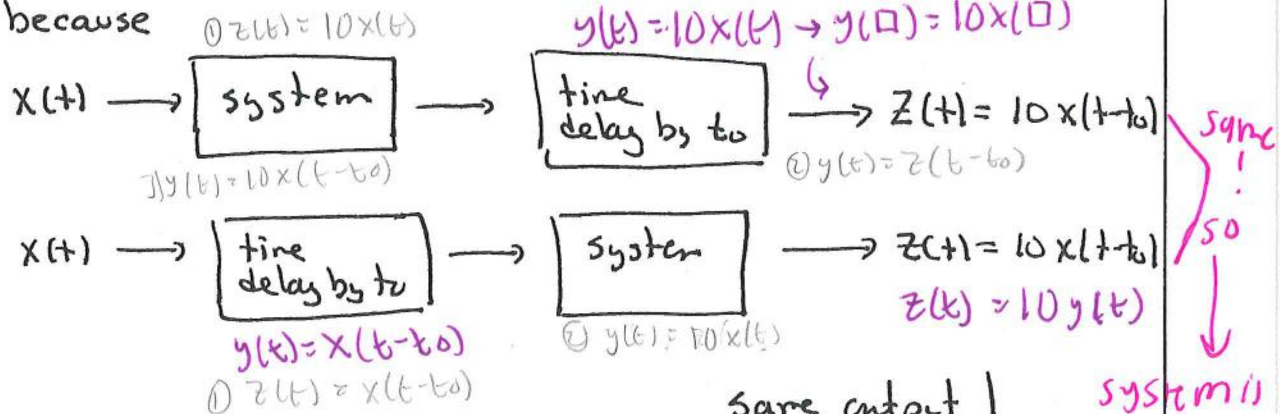
DT

$$x[n-n_0] \rightarrow \boxed{\text{system}} \rightarrow y[n-n_0]$$

Definition #4 : A system is called "time-invariant" if for any input signal $x(t)$ $(x[n])$ and for any $t_0 \in \mathbb{R}$ $(n_0 \in \mathbb{Z})$, the systems output when the input is shifted $x(t-t_0)$ $(x[n-n_0])$ is the shifted output $y(t-t_0)$ $(y[n-n_0])$.

Example 1: The system defined by $y(t) = 10\,x(t)$ is time-invariant.

because



same output !

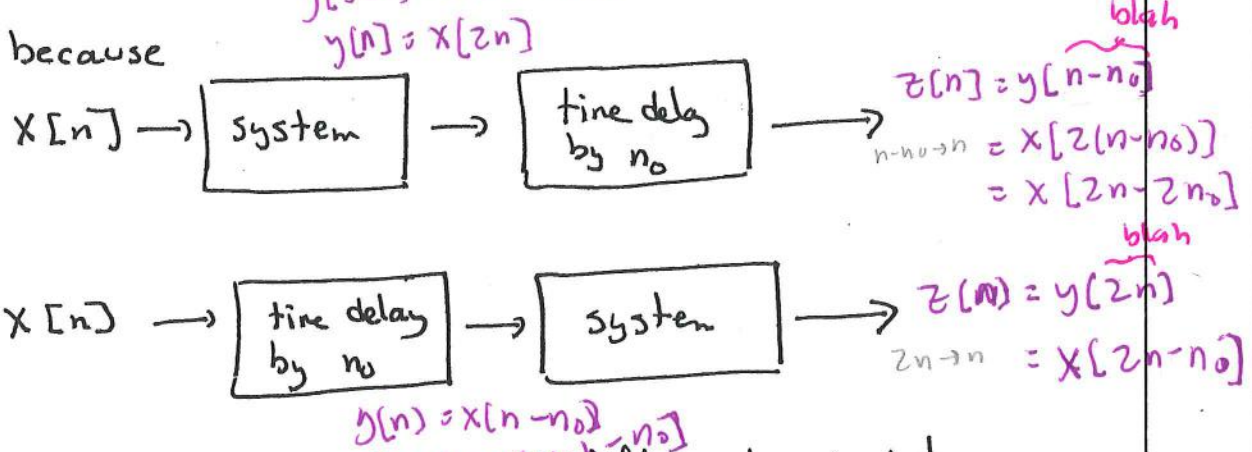
system is time invariant

Example 2: The system defined by $y[n] = x[2n]$ is not time-invariant

$y[\text{blah}] = x[2\cdot\text{blah}]$

$y[n] = x[2n]$

because



different output !

different outputs so system is ~~NOT~~ NOT T.I.

$y[n] = f(x[n])$

$y(t) = f(x(t))$

$t\,x(t),\ (n-1)^2 x[n]$

Part I

6.8

We are particularly interested in "LTI systems"

= linear and time-invariant systems.

Exercises: Which of these systems are LTI?

1. $y[n] = x[n-1]$
2. $y(t) = x(-t)$
3. $y(t) = t\,x(t)$
4. $y(t) = x(t+3) - x(t-3)$
5. $y[n] = x[n] + n$
6. $y[n] = \mathrm{Re}(x[n])$
7. $y(t) = |x(t)|$
8. $y[n] = \frac{x[n-1] + x[n] + x[n+1]}{3}$
9. $y(t) = \frac{1}{6}\int_{t-3}^{t+3} x(\tau)\,d\tau$
10. $y(t) = \frac{d}{dt}x(t)$
11. $y[n] = x[n] - x[n-1]$

Part I

7. CT and DT convolution

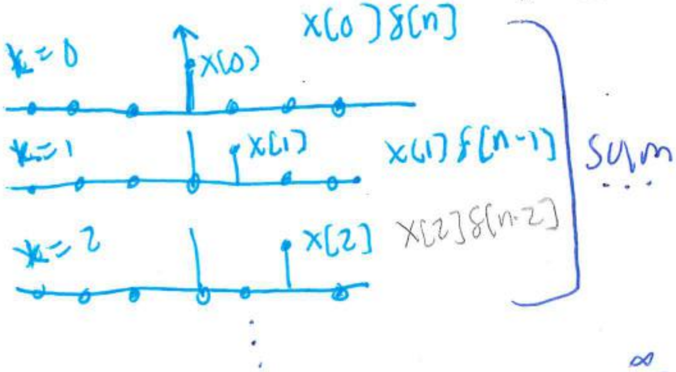
a) convolution sum and DT LTI systems

b) convolution integral and CT LTI systems

a) Convolution Sum and DT LTI Systems

observe: Any DT signal can be written as a linear combination of shifted unit impulses

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k]$$



$(\{\delta[n-k]\}_{k\in\mathbb{Z}}$ is a basis)

by linearity

$$\delta[n-k] \rightarrow \boxed{\text{system}} \rightarrow h_k[n]$$

$$x[k]\,\delta[n-k] \rightarrow \boxed{\text{system}} \rightarrow x[k]\,h_k[n]$$

$$\sum_{k=-\infty}^{\infty} x[k]\,\delta[n-k] \rightarrow \boxed{\text{system}} \rightarrow \sum_{k=-\infty}^{\infty} x[k]\,h_k[n]$$

Corollary #1: The reponse of a DT linear system can be written as a sum

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\,h_k[n]$$

where $h_k[n]$ is system's response to $\delta[n-k]$.

unit impulse response

if $\delta[n] \rightarrow \boxed{\text{system}} \rightarrow h(n]$ and system is T.I.

unit input impulse

then $\delta[n-k] \rightarrow \boxed{} \rightarrow h_k[n] = h[n-k]$

$$x[n]\,\delta[n-k] \rightarrow \boxed{} \rightarrow x(k]\,h_k[n]$$

$$\sum_{k} x(k]\,\delta[n-k] \rightarrow \boxed{} \rightarrow \sum_{k} x[k]\,h_k[n]$$

memorize!

→

ON EXAM

Corollary #2: The response of a DT $\underline{\underline{\text{LTI}}}$ system as convolution

can be written as a sum $= \sum_{-\infty}^{\infty} x[k]\, h_k[n]$

$$\underset{\sim\sim}{y[n]} = \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

$x[n] * h[n]$

where h[n] is the system's response to $\delta[n]$.

↑

"unit impulse response"

Definition: The "convolution" $*$ between two DT signals $x_1[n]$ and $x_2[n]$ is the sum

$$x_1[n] * x_2[n] = \sum_{k=-\infty}^{\infty} x_1[k]\, x_2[n-k]$$

The output of an LTI system is the convolution of the input x[n] with the unit impulse response h[n] of the system

$$y[n] = x[n] * h[n]$$

Part I

$$X[n] \longrightarrow \overset{\text{DT system}}{\boxed{\quad h[n] \quad}} \longrightarrow y[n] = x[n] * h[n]$$

$$= \sum_{k} \overset{x[k]}{x[\cancel{k}]}\, h[n-k]$$

where

$$\delta[n] \longrightarrow \boxed{\qquad} \longrightarrow h[n]$$

★ on test

Example 1: The unit impulse response of an LTI system is

$$h[n] = \delta[\underline{n-3}].$$

Compute the system's response to the signal

$$x[n] = 2^{-n} u[n].$$

$$2^{-n}u[n] \rightarrow \boxed{h[n]=\delta[n-3]} \rightarrow ?$$

$$y[n] = x[n] * h[n]$$

$$= \sum_{k=-\infty}^{\infty} x[k]\, h[\underset{\uparrow}{n}-k]$$

$$= \sum_{k=-\infty}^{\infty} 2^{-k} u[k]\, \overbrace{\delta[\underline{n-k}\,-\underline{3}]}^{=0 \text{ except for when } n-k-3=0 \Leftrightarrow k=n-3} \qquad (\text{see } ★)$$

$$= \sum_{k=-\infty}^{\infty} 2^{-k} u[k] \begin{cases} 0, & \text{if } k \neq n-3 \\ 1, & \text{if } k = n-3 \end{cases}$$

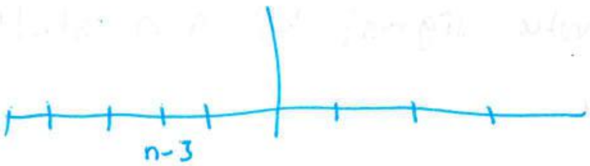
$$= 2^{-(n-3)} u[n-3]$$

$$= 2^{-n+3} u[n-3]$$

★ $y[n] = h[n] * x[n]$ shown on next page

★ $\delta[n-k-3] = 0$ everywhere except $k = n-3$

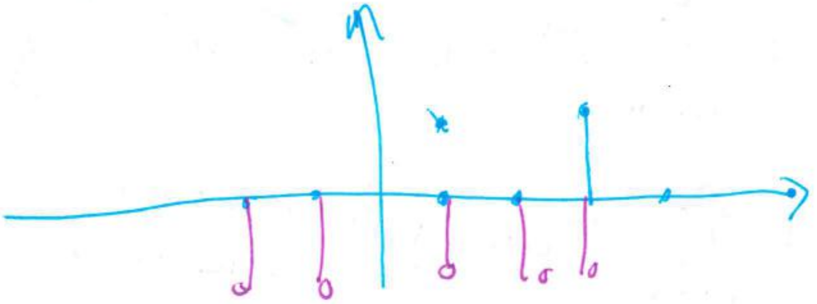
so $f[k]\,\delta[n-k-3] = f[n-3]\,\delta[n-k-3]$



$$\text{so } \sum_{k=-\infty}^{\infty} \underbrace{2^{-k}u[k]}_{f[k]}\,\delta[n-k-3] = \sum^{\infty} 2^{-(n-3)} u[n-3]\,\delta[n-k-3]$$

$$= 2^{-(n-3)} u[n-3] \sum_{k=-\infty}^{\infty} \delta[n-k-3]$$

$$= 2^{-n+3} u[n-3]$$



→ if you convolve signal w/ non-shifted delta

Example 2: The unit impulse response of an LTI system is

$$h[n] = u[n].$$

Compute the system's response to the input

$$x[n] = 2^{-n} u[n].$$

$h[n] = \delta[n-3]$

$x[n] = 2^{-n} u[n]$

$$y[n] = h[n] * x[n]$$

$$= \sum_{k=-\infty}^{\infty} h[k]\, x[n-k] = \sum_{k=-\infty}^{\infty} \underbrace{\delta[k-3]}_{\delta[k-3] = 0 \text{ for all } k \text{ except } k=3} \overbrace{2^{-n+k}\, u[n-k]}^{f[x] \to f[3]}$$

$\underline{f[x] \to f[3]}$

$$= \sum_{k=-\infty}^{\infty} \delta[k-3]\, 2^{-n+3} u[n-3] = 2^{-n+3} u[n-3] \sum_{k=-\infty}^{\infty} \delta[n-3]$$

$$= 2^{-n+3} u[n-3]$$

[illegible]

→ ✱ EXAMPLE 2 shown on nb paper on next page!

In general

$$x[n] * \delta[n] = x[n]$$

$$x[n] * \delta[n-n_0] = x[n-n_0]$$

time delay by n_0 is LTI

$$x[n] \rightarrow \boxed{h[n] = \delta[n-h_0]} \rightarrow y(n)$$

Example 1: The unit impulse response of an LTI system is

$$h(t) = \delta(t-3).$$

Compute the system's response to the input

$$x(t) = e^{-t}u(t).$$

$$e^{-t}u(t) \longrightarrow \boxed{\delta(t-3)} \longrightarrow ?$$

$$x(t-3) = e^{-(t-3)}u(t-3) = e^{-t+3}u(t-3)$$

$$y(t) = x(t) * h(t)$$

$$= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau$$

$$= \int_{-\infty}^{\infty} \underbrace{e^{-\tau}u(\tau)}_{f(\tau)\to f(t-3)}\,\underbrace{\delta(t-\tau-3)}_{=0 \text{ everywhere (all } \tau)}\,d\tau$$

except when $t-\tau-3=0$

$t-3=\tau$

shifting property of $\delta(t)$

$$= \int_{-\infty}^{\infty} e^{-(t-3)}u(t-3)\,\delta(t-\tau-3)\,d\tau$$

$$= e^{-t+3}u(t-3)\int_{-\infty}^{\infty}\delta(t-\tau-3)\,d\tau$$

$$= e^{-t+3}u(t-3)$$

In general

$$x(t) * \delta(t-t_0) = x(t-t_0)$$

$$x(t) * \delta(t) = x(t)$$

$$y(t) = 3x(t-t_0) + 2x(t)$$

$$h(t) = 3\delta(t-t_0) + 2\delta(t)$$

→ to obtain $h(t)$, replace $x(t)$ by $\delta(t)$ in the expression for $y(t)$

T.D. by t_0

$$x(t) \longrightarrow \boxed{h(t) = \delta(t-t_0)} \longrightarrow y(t) = x(t-t_0)$$

Example 2: The unit impulse response of an LTI system is

$$h(t) = e^{-2t}u(t).$$

Compute the system's response to the input

$$x(t) = u(t).$$

$$u(t) \longrightarrow \boxed{e^{-2t}u(t)} \longrightarrow ?$$

$$y(t) = x(t) * h(t)$$

* know by ♡ -exam

$$\left[= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau\right] = \int_{-\infty}^{\infty} u(\tau)\, e^{-2(t-\tau)}\, u(t-\tau)\, d\tau$$

$$= \int_{0}^{\infty} e^{-2(t-\tau)} \underbrace{u(t-\tau)}_{\text{zero when } \tau > t}\, d\tau$$

but $u(t-\tau) = \begin{cases} 1, & \text{if } t-\tau \geq 0 \\ 0, & \text{else} \end{cases}$

so $y(t) = \begin{cases} \int_0^t e^{-2(t-\tau)}\, d\tau, & \text{if } t \geq 0 \\ 0, & \text{else} \end{cases}$

so...

$$y(t) = u(t)\int_0^t e^{-2t} e^{2\tau}\, d\tau = u(t)\, e^{-2t} \int_0^t e^{2\tau}\, d\tau$$

$$= u(t)\, e^{-2t} \left.\frac{e^{2\tau}}{2}\right|_0^t$$

$$= u(t)\, e^{-2t} \left[\frac{e^{-2t} - e^{0}}{2}\right]$$

$$= \frac{u(t)}{2}(1 - e^{-2t})$$

* on test

* geometric series

Example 2

$h[n] = u[n]$

$x[n] = 2^{-n} u[n]$

P 7.4

Ex 2)

$$2^{-n} u[n] \longrightarrow \boxed{h[n] = u[n]} \rightarrow ?$$

$y[n] = x[n] * h[n]$

zero points if not right on test

$$= \sum_{k=-\infty}^{\infty} x[k]\, h[n-k]$$

$$= \sum_{k=-\infty}^{\infty} 2^{-k} u[k] \; u[n-k]$$

but

$$u[k] = \begin{cases} 1, \text{ if } k \geq 0 \\ 0, \text{ if } k < 0 \end{cases}$$

so..

$$y[n] = \sum_{k=0}^{\infty} 2^{-k} u[n-k]$$

$n - k \geq 0$

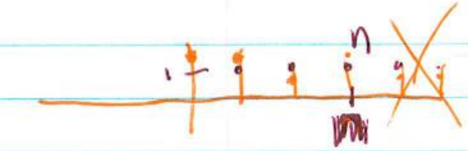
$n \geq k$

$$u[n-k] = \begin{cases} 1, \text{ if } n-k \geq 0 \text{ .... so } k \leq n \\ 0, \text{ else} \end{cases}$$

so

$$y[n] = \begin{cases} \sum_{k=0}^{n} 2^{-k}, \text{ if } n \geq 0 \\ 0, \text{ if } n < 0 \end{cases}$$

* ppl always forget!



so.. $\sum_{k=0}^{\infty} 2^{-k} \begin{cases} 1, \text{ if } k \leq n \\ 0, \text{ else} \end{cases}$

$\left(\frac{1}{2}\right)^k$

$$y[n] = \begin{cases} \frac{1 - \left(\frac{1}{2}\right)^{n+1}}{1 - \frac{1}{2}}, \text{ if } n \geq 0 \\ 0, \text{ else} \end{cases}$$

$$= \begin{cases} 2 - \left(\frac{1}{2}\right)^n, \text{ if } n \geq 0 \\ 0, \text{ else} \end{cases} = \left(2 - \left(\tfrac{1}{2}\right)^n\right) u[n]$$

b) Convolution integral and CT LTI systems

observe : Any CT signal can be written as an integral of shifted unit impulses

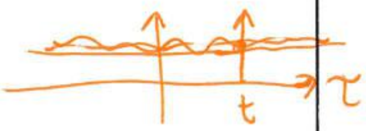
$$x(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau$$

$$x(t) = \int x(\tau)\,\delta(t-\tau)\,d\tau$$

Why?

for any t, $x(\tau)\overbrace{\delta(t-\tau)}^{\text{zero for all } t \neq \tau} \overset{\infty \text{ if } t=\tau}{=} x(t)\,\delta(t-\tau)$

$$\int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau = \int_{-\infty}^{\infty} x(t)\,\delta(t-\tau)\,d\tau$$



$$= x(t)\int_{-\infty}^{\infty} \delta(t-\tau)\,d\tau = x(t)\cdot 1$$

$$= x(t)$$ ✓ proof

Corollary #1 : The response of a CT linear system can be written as an integral

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h_{\tau}(t)\, d\tau$$

where $h_{\tau}(t)$ is the system's response to $\delta(t-\tau)$.

if

$$\delta(t-\tau) \longrightarrow \boxed{\phantom{\text{linear}}} \longrightarrow h_{\tau}(t)$$

then

$$x(\tau)\,\delta(t-\tau) \longrightarrow \boxed{\text{linear}} \longrightarrow x(\tau)\, h_{\tau}(t)$$

also

$$\int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\, d\tau \longrightarrow \boxed{\text{linear}} \longrightarrow \int_{-\infty}^{\infty} x(\tau)\, h_{\tau}(t)\, d\tau$$

Corollary #2: The response of a CT LTI system can be written as an integral

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

where $\underset{\substack{\uparrow \\ \text{"unit impulse response"}}}{h(t)}$ is the system's response to $\delta(t)$.

$$\delta(t-\tau) \rightarrow \boxed{T.I.} \rightarrow h_\tau(t) = h(t-\tau)$$

$$\delta(t) \rightarrow \boxed{} \rightarrow h(t) \quad \text{* unit impulse response}$$

So if system is linear + time invariant

$$y(t) = \int_{-\infty}^{\infty} x(\tau) \underbrace{h_\tau(t)}_{h(t-\tau)} d\tau = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

Definition: The "convolution" * between two CT signals $x_1(t)$ and $x_2(t)$ is the integral

$$\left[x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau)\, x_2(t-\tau)\, d\tau \right]$$

The output of an LTI system is the convolution of the input $x(t)$ with the unit impulse response $h(t)$ of the system

$$y(t) = x(t) * h(t)$$

1. Properties of LTI systems

CT

DT

① $x(t) \rightarrow \boxed{h(t)} \rightarrow y(t) = x(t) * h(t)$

$x[n] \rightarrow \boxed{h[n]} \rightarrow y[n] = x[n] * h[n]$

② $x(t) \rightarrow \boxed{h(t)} \rightarrow y(t)$

same output as

$h(t) \rightarrow \boxed{x(t)} \rightarrow y(t)$

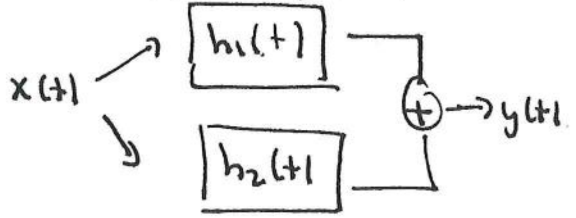
$x[n] \rightarrow \boxed{h[n]} \rightarrow y[n] = x[n] * h[n]$

same output as

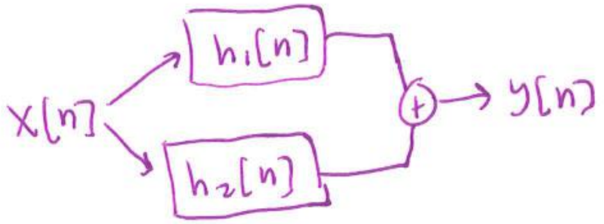
$h[n] \rightarrow \boxed{x(n)} \rightarrow y(n) = h(n) * x[n)$

③ $x(t) \rightarrow \boxed{h_1(t) + h_2(t)} \rightarrow y(t)$

same output as

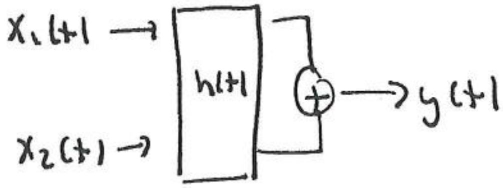


$x[n] \longrightarrow \boxed{h_1[n] + h_2[n]} \rightarrow y[n]$



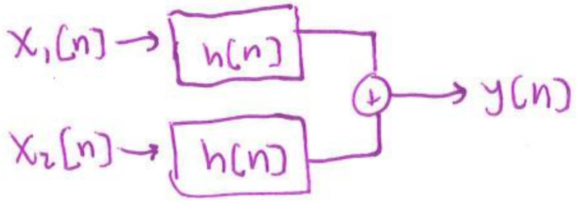
④ $x_1(t) + x_2(t) \rightarrow \boxed{h(t)} \rightarrow y(t)$

same output as



$x_1[n] + x_2(n] \rightarrow \boxed{h[n]} \rightarrow y(n]$

same output as



⑤ $x(t) \rightarrow \boxed{h_1(t)} \rightarrow \boxed{h_2(t)} \rightarrow y(t)$

same output as

$x(t) \rightarrow \boxed{h_1(t) * h_2(t)} \rightarrow y(t).$

$x[n] \rightarrow \boxed{h_1[n]} \rightarrow \boxed{h_2(n]} \rightarrow y(n)$

same output as

$x[n] \rightarrow \boxed{h_1[n] * h_2(n]} \rightarrow y(n)$

✱ Could be on test (proofs of properties)

Justification for Property ②: by community of *

math proof of commutivity

$$x_1[n] * x_2[n] = \sum_{k=-\infty}^{\infty} x_1[k]\, x_2[\overbrace{n-k}^{k'}]$$

let $k' = n-k$

$$= \sum_{k'=-\infty}^{\infty} x_1[n-k']\, x_2[k']$$

$$= \sum_{k=-\infty}^{\infty} x_2[k]\, x_1[n-k]$$

$$= x_2[n] * x_1[n]$$

Justification for Property ③:

why? bc of distributinity of *

$$x[n] * \left(h_1[n] + h_2[n]\right) = \sum_{k=-\infty}^{\infty} x[k]\left(h_1[n-k] + h_2[n-k]\right)$$

$$= \sum_{k=-\infty}^{\infty} \left(x[k]\, h_1[n-k] + x[k]\, h_2[n-k]\right)$$

$$= \sum_{k=-\infty}^{\infty} x[k]\, h_1[n-k] + \sum_{k=-\infty}^{\infty} x[k]\, h_2[n-k]$$

$$= x[n] * h_1[n] + x[n] * h_2[n]$$

Justification for Property ④ :

why? bc $*$ is commutative & distributive

$$(x_1[n] + x_2[n]) * h[n] = h[n] * (x_1[n] + x_2[n])$$

by commutativity

$$= h[n] * x_1[n] + h[n] * x_2[n]$$

by distributivity

IDK!

Justification for Property ⑤ :

why? bc of associativity of $*$

$$(x_1[n] * x_2[n]) * x_3[n] = \Big(\underbrace{\sum_{k=-\infty}^{\infty} x_1[k]\, x_2[n-k]}_{\leftarrow f[n]}\Big) * x_3[n]$$

let $m' = m-k$

$m = m'+k$

$$= \sum_{m'=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x_1[k]\, x_2[m']\, x_3[n-m'-k] = \sum_{m=-\infty}^{\infty} \underbrace{\sum_{k=-\infty}^{\infty} x_1[k]\, \overparen{x_2[m-k]}}_{f[m]} x_3[n-m]$$

replace
m' by k
k by m

& change of variables

$$= \sum_{m=-\infty}^{\infty} x_1[m] \sum_{k=-\infty}^{\infty} \overparen{x_2[k]}\, x_3[\overset{n-m-k}{\text{[illegible]}}-k]$$

$$= x_1[n] + \Big(\sum_{k=-\infty}^{\infty} x_2[k]\, x_3[n-k]\Big)$$

$$= x_1[n] * (x_2[n] * x_3[n])$$

$\uparrow$

Additional Properties of LTI systems.

For Memoryles LTI systems

CT	DT
$h(t) = K\delta(t),\ K \in \mathbb{C}$	$h[n] = K\delta[n],\ K \in \mathbb{C}$
$y(t) = K\,x(t)$	$y[n] = K\,x[n]$

Why? bc $y[n] = x[n] * h[n]$

$$= \sum_{k=-\infty}^{\infty} x[k]\,h[n-k]$$

so $y[n]$ will depend on $x[n+1]$ if $h[n-(n+1)] \neq 0$
$h[-1] \neq 0$

$x[n+2]$ if $h[n-(n+2)] \neq 0$
$h[-2] \neq 0$

$x[n+3]$ if $h[n-(n+3)] \neq 0$
$h[-3] \neq 0$

$\vdots$

$x[n-1]$ if $h[n-(n-1)] \neq 0$
$h[1] \neq 0$

$x[n-2]$ if $h[n-(n-2)] \neq 0$
$h[2] \neq 0$

$x[n-3]$ if $h[n-(n-3)] \neq 0$
$h[3] \neq 0$

so for $y[n]$ to not depend on $x[n+1], x[n+2], \ldots$ need $h[-1], h[-2], h[-3], \ldots = 0$

and for $y[n]$ to not depend on $x[n-1], x[n-2] \ldots$ need $h[1], h[2], h[3] \ldots = 0$

∴ the only non-zero $h[n]$ will be $h[0] = k \rightarrow h[n] = k\delta[n]$

For Invertible LTI systems

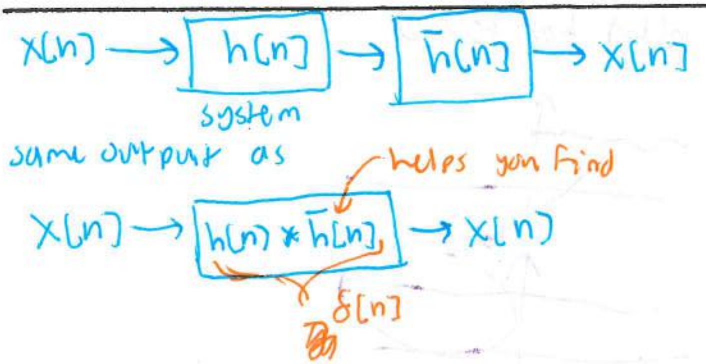
CT

If $h(t)$ is unit impulse response of system and $\bar{h}(t)$ is unit impulse response of inverse system, then

$$\rightarrow h(t) * \bar{h}(t) = \delta(t)$$

DT

same but w/ $h[n]$



For Causal LTI systems

CT	DT
$h(t)=0$ for $t<0$	$h[n]=0$ for $n<0$
$y(t)=\int_{-\infty}^{t} x(\tau)h(t-\tau)d\tau$	$y[n]=\sum_{k=-\infty}^{n} x[k]\,h[n-k]$

$h[n-k]=0$ for $n-k<0$

$k>n$

$y(t)=\int_{-\infty}^{t} x(\tau)h(t-\tau)d\tau$ $\quad$ $y[n]=\sum_{k=-\infty}^{n} x[k]h[n-k]$

For stable LTI systems *finite (real num)

CT

$$\int_{-\infty}^{\infty} |h(t)|\,dt \text{ is finite}$$

DT

$$\sum_{n=-\infty}^{\infty} |h[n]| \text{ is finite.}$$

⇒ ? go fix mistake!
wrong!

↳ Show that $\sum_{n=-\infty}^{\infty} |h[n]|$ if finite ⇒ system is stable

Assume $x[n]$ is bounded

so there exists m such that $|x[m]| < m$

We have

$a+b \geq c$

$$|y[n]| = |h[n] * x[n]|$$

$$= \left|\sum_{k=-\infty}^{\infty} h[k]\,x[n-k]\right| \leq \sum_{k=-\infty}^{\infty} |h[k]\,x[n-k]|,$$

by Δ inequality

$$= \sum_{k=-\infty}^{\infty} |h[k]|\overbrace{|x[n-k]|}^{<m}$$

$$< \sum_{k=-\infty}^{\infty} |h[k]|\,m$$

$$= m\underbrace{\sum_{k=-\infty}^{\infty} |h[k]|}$$

so if this sum is less than K

$$\Rightarrow |y[n]| < \underbrace{mK}_{M}$$

$$e^{j2\pi 440t} \longrightarrow \boxed{} \longrightarrow H(2\pi 440)\,e^{j2\pi 440t}$$

$\uparrow$

makes it louder or
less loud but frequency
is the same

$$Z^n \longrightarrow \boxed{} \longrightarrow \overset{\substack{\text{eigenvalue}\\ \downarrow}}{H}(Z)\boxed{Z^n} \text{ eigen factor}$$

$$(1+3j)^n \longrightarrow \boxed{} \longrightarrow \underbrace{H(1+3j)}_{\text{number}}{}^{n}\,(1+3j)^n$$

$$e^{jwn} \longrightarrow \boxed{} \longrightarrow \underbrace{H(e^{jw})\,e^{jwn}}_{H(w)}$$

$H(w)$ freq response

$H(z)$ transfer function

$H(w) = H(e^{jw})$