AC-1 August 2016 QE

1. Without performing any block-diagram reduction,

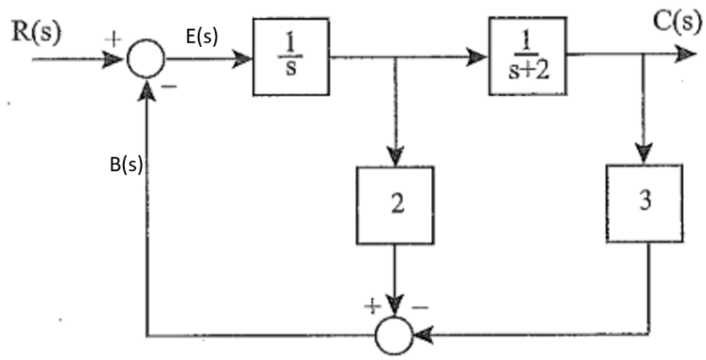

Figure 1: Problem 1 block diagram.

(a) determine the feedforward transfer function.
Feedforward transfer function $TF_{FF} = \frac{C(s)}{E(s)}$.

$$C(s) = \frac{E(s)}{s(s+2)} \Rightarrow \frac{C(s)}{E(s)} = \frac{1}{s(s+2)}$$

(b) determine the open-loop transfer function.
Open-loop transfer function $TF_{OL} = \frac{B(s)}{E(s)}$.

$$B(s) = \frac{2E(s)}{s} - \frac{3E(s)}{s(s+2)} = E(s)\left(\frac{2}{s} - \frac{3}{s(s+2)}\right) \Rightarrow \frac{B(s)}{E(s)} = \frac{2}{s} - \frac{3}{s(s+2)} = \frac{2s+1}{s(s+2)}$$

(c) determine the closed-loop transfer function.
Closed-loop transfer function

$$TF_{CL} = \frac{C(s)}{R(s)} = \frac{TF_{FF}}{1+TF_{OL}} = \frac{\frac{1}{s(s+2)}}{1+\frac{2s+1}{s(s+2)}} = \frac{1}{s^2+4s+1}$$

2. Given the RLC circuit shown

(a) Obtain a block diagram of the system with $E_{in}(s)$ as input and $I_L(s)$ as output.

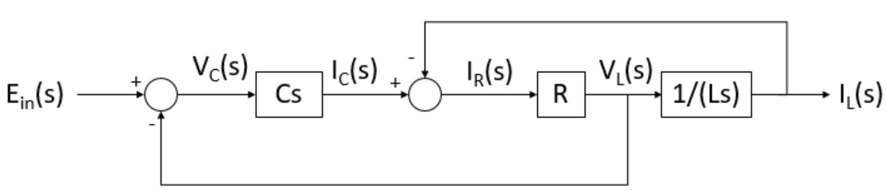

Figure 2: Problem 2 block diagram.

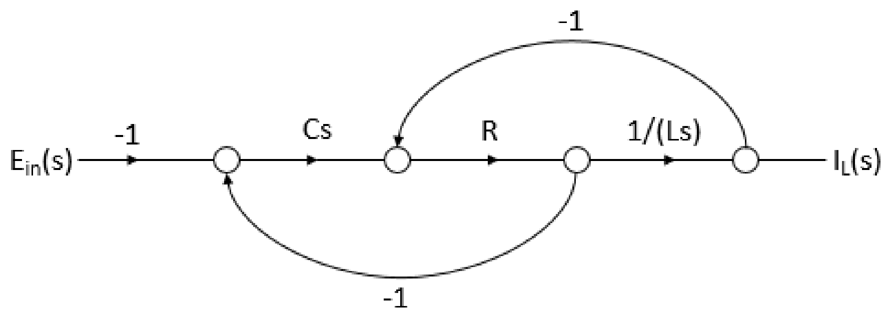

Figure 3: Problem 2 signal flow graph.

(b) Find the overall transfer function $\frac{I_L(s)}{E_{in}(s)}$ from your block diagram using Mason's Gain Formula. Express the transfer function in terms of R, L, C components only.

Mason's Gain Formula:

$$\frac{C(s)}{R(s)} = \frac{\sum_k P_k \Delta_k}{\Delta},$$

where P_k is the path gain of the kth path, $\Delta = 1 - \sum_i L_i + \sum_{i,j} L_i L_j - \cdots$, and $\Delta_k = 1 -$ (non-touching loops).

$$P_1 = \frac{RCs}{Ls} = \frac{RC}{L}, \qquad \Delta_1 = 1, \qquad L_1 = -RCs, \qquad L_2 = -\frac{R}{Ls}$$

$$\Delta = 1 - (L_1 + L_2) = 1 + RCs + \frac{R}{Ls}$$

$$\frac{I_L(s)}{E_{in}(s)} = \frac{P_1 \Delta_1}{\Delta} = \frac{\frac{RC}{L}}{1 + RCs + \frac{R}{Ls}} = \frac{RCs}{RLCs^2 + Ls + R}.$$

3. Given a negative unity feedback control system with compensator $G_c(s)$ and plant $G(s) = \frac{9}{(s+1)(s+3)}$. It is desirable to place the dominant closed-loop poles of the compensated system at $s_1^*, s_2^* = -3 \pm j4$.

(a) Determine the angle of deficiency ϕ to the dominant closed-loop pole at $s_1^* = -3 \pm j4$.
σ-axis intercept, σ_I, and angles of asymptotes, θ_A:

$$\sigma_I = \frac{\sum \text{poles} - \sum \text{zeros}}{\#\text{poles} - \#\text{zeros}} = \frac{(-1-3)}{2} = -2, \qquad \theta_A = \frac{(2k+1)180^\circ}{\#\text{poles} - \#\text{zeros}} = 90^\circ + k180^\circ = \pm 90^\circ$$

Angle of deficiency ϕ:

$$\phi = 180^\circ - \sum(\text{angles of vectors to } s_1^* \text{ from poles in } G(s)) = 180^\circ - \left[90^\circ + \left(180^\circ - \arctan\left(\frac{4}{2}\right)\right)\right] = 26.565^\circ$$

(b) A student decides to design a lead compensator such that the compensated system $G_d(s)$ will have the dominant closed-loop poles at $s_1^*, s_2^* = -3 \pm j4$. The student decided to place the zero to cancel out one of the poles of $G(s)$. What is the resulting transfer function $G_c(s)$?

Let $z_c = -3$ to cancel out the $s = -3$ pole of $G(s)$. Then we want $\sigma_I = -3$, so

$$\sigma_I = -3 = \frac{-1 + p_c}{3 - 1} \quad \Rightarrow \quad p_c = -5 \qquad \text{and} \qquad G_c(s) = \frac{K_c(s+3)}{(s+5)}.$$

Find K_c:

$$G_c(s)G(s) = K_c\left(\frac{s+3}{s+5}\right)\left(\frac{9}{(s+1)(s+3)}\right) = \frac{9K_c}{(s+1)(s+5)}$$

$$\left|\frac{9K_c}{(s+1)(s+5)}\right|_{s=-3+j4} = \left|\frac{9K_c}{(-3+j4+1)(-3+j4+5)}\right| = \left|\frac{9K_c}{(-2+j4)(2+j4)}\right| = 1$$

$$\Rightarrow 9K_c = 20 \qquad K_c = \frac{20}{9}$$

$$G_c(s) = \frac{20(s+3)}{9(s+5)}.$$

(c) After the student has designed the above lead compensator, he/she found out that the resultant K_v value is too small and would like to increase the $K_v^{G_d}$ by 10 times. The student has decided to use a lag compensator $G_c(s) = \frac{K_c(s+z_1)}{(s+p_1)}$ for it. Determine z_1 and p_1 of the lag compensator. How can the student verify that his/her lag compensator design is satisfactory?
Let $z_1 = 0.05$ and $p_1 = 0.005$, so we have

$$G_c(s) = \frac{K_c(s+0.05)}{(s+0.005)}.$$

Then the new open-loop transfer function is

$$G_c(s)G(s) = \left(\frac{s+0.05}{s+0.005}\right)\left(\frac{20}{(s+1)(s+5)}\right),$$

and the value of $K_v^{G_d}$ is increased by a factor of 10. To verify the design, the student can check the angle from one of the desired dominant closed-loop poles.

4. Given a negative unity feedback control system with $G(s) = \frac{K+4s}{s(s^2+8s+16)}$,

(a) Sketch the root locus for $K > 0$.
First, obtain the characteristic equation $1 + G(s)H(s) = 0$.

$$1 + \frac{K+4s}{s(s^2+8s+16)} = \frac{s^3+8s^2+16s+4s+K}{s(s^2+8s+16)} = s^3+8s^2+20s+K = 0$$

$$\Rightarrow s^3+8s^2+20s+K = 1 + \frac{K}{s(s^2+8s+20)} = 0$$

Thus, $p_1 = 0$ and $p_{2,3} = -4 \pm j2$. Then $\sigma_I = -\frac{8}{3}$ gives us the intersection of the asymptotes with the σ-axis and $\theta_A = \pm 60°, 180°$ gives us the angles of the asymptotes. Finally, the breakaway and break-in points need to be found.

$$\sum \frac{1}{\sigma + z_i} = \sum \frac{1}{\sigma + p_i} \text{ for real } \sigma$$

$$\Rightarrow \frac{1}{\sigma+4+j2} + \frac{1}{\sigma+4-j2} + \frac{1}{\sigma} = 3\sigma^2 + 16\sigma + 20 = 0$$

$$\Rightarrow \sigma = -\frac{10}{3}, -2.$$

The root locus sketch is as in Fig. 4.

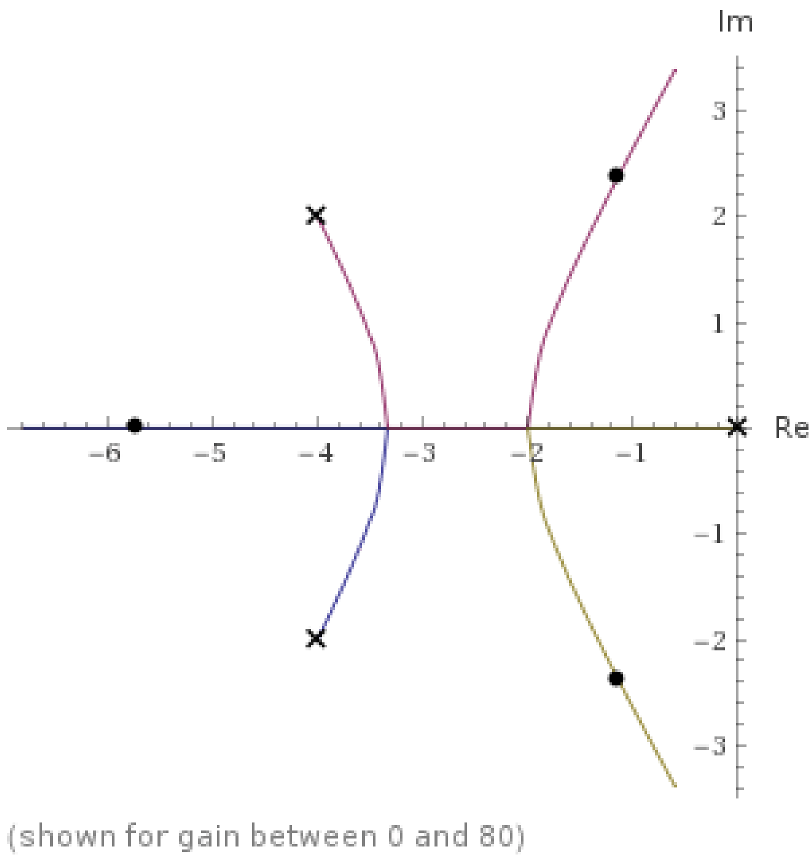

Figure 4: Problem 4 root locus plot (plotted on wolframalpha.com).

(b) Determine the angle of departure/arrival, if any.

Angle of departure from a complex pole, θ_{p_i}:

$$\theta_{p_i} = 180^\circ - (\text{sum of the angles of vectors to complex pole i from other poles}) + (\text{sum of the angles of vectors to complex pole i from zeros})$$

$$\theta_{p_1} = 180^\circ - [(360^\circ - \arctan(0.5)) + \arctan(0.5)] = -180^\circ.$$

$$\theta_{p_2} = 180^\circ - [90^\circ + (180^\circ - \arctan(0.5))] = -90^\circ + 26.57^\circ = 63.43^\circ.$$

$$\theta_{p_3} = 180^\circ - [-90^\circ + (180^\circ + \arctan(0.5))] = 90^\circ + 206.57^\circ = -63.43^\circ.$$

(c) Determine the value of K and the frequency at which the loci cross the $j\omega$-axis, if any.

s^3	1	20
s^2	8	K
s^1	$\frac{160-K}{8}$	
s^0	K	

$$\frac{160-K}{8} = 0 \quad \Rightarrow \quad K = 160 \text{ at the crossing point}$$

$$8s^2 + 160 = 0 \quad \Rightarrow \quad s^2 = -20 \quad \Rightarrow \quad s = j\omega = \pm j2\sqrt{5} \quad \Rightarrow \quad \omega = \pm 2\sqrt{5}.$$

The frequencies at which the loci cross the $j\omega$-axis are $\omega = \pm 2\sqrt{5}$ when $K = 160$.