filtering in Chapters 3–5 and in our examination of systems described by linear constant-coefficient differential or difference equations, and we will gain a further appreciation for its utility in Chapter 6, in which we examine filtering and time-versus-frequency issues in more detail. In addition, the multiplication properties in continuous and discrete time are essential to our development of sampling in Chapter 7 and communications in Chapter 8.

Chapter 5 Problems

The first section of problems belongs to the basic category and the answers are provided in the back of the book. The remaining three sections contain problems belonging to the basic, advanced, and extension categories, respectively.

BASIC PROBLEMS WITH ANSWERS

5.1. Use the Fourier transform analysis equation (5.9) to calculate the Fourier transforms of:

(a) $(\frac{1}{2})^{n-1}u[n-1]$ **(b)** $(\frac{1}{2})^{|n-1|}$

Sketch and label one period of the magnitude of each Fourier transform.

5.2. Use the Fourier transform analysis equation (5.9) to calculate the Fourier transforms of:

(a) $\delta[n-1]+\delta[n+1]$ **(b)** $\delta[n+2]-\delta[n-2]$

Sketch and label one period of the magnitude of each Fourier transform.

5.3. Determine the Fourier transform for $-\pi \leq \omega < \pi$ in the case of each of the following periodic signals:

(a) $\sin(\frac{\pi}{3}n+\frac{\pi}{4})$ **(b)** $2+\cos(\frac{\pi}{6}n+\frac{\pi}{8})$

5.4. Use the Fourier transform synthesis equation (5.8) to determine the inverse Fourier transforms of:

(a) $X_1(e^{j\omega}) = \sum_{k=-\infty}^{\infty}\{2\pi\delta(\omega-2\pi k)+\pi\delta(\omega-\frac{\pi}{2}-2\pi k)+\pi\delta(\omega+\frac{\pi}{2}-2\pi k)\}$

(b) $X_2(e^{j\omega}) = \begin{cases} 2j, & 0<\omega\leq\pi \\ -2j, & -\pi<\omega\leq 0 \end{cases}$

5.5. Use the Fourier transform synthesis equation (5.8) to determine the inverse Fourier transform of $X(e^{j\omega}) = |X(e^{j\omega})|e^{j\sphericalangle X(e^{j\omega})}$, where

$$|X(e^{j\omega})| = \begin{cases} 1, & 0\leq|\omega|<\frac{\pi}{4} \\ 0, & \frac{\pi}{4}\leq|\omega|\leq\pi \end{cases} \quad \text{and } \sphericalangle X(e^{j\omega}) = -\frac{3\omega}{2}.$$

Use your answer to determine the values of n for which $x[n]=0$.

5.6. Given that $x[n]$ has Fourier transform $X(e^{j\omega})$, express the Fourier transforms of the following signals in terms of $X(e^{j\omega})$. You may use the Fourier transform properties listed in Table 5.1.

(a) $x_1[n] = x[1-n]+x[-1-n]$

(b) $x_2[n] = \frac{x^*[-n]+x[n]}{2}$

(c) $x_3[n] = (n-1)^2x[n]$

5.7. For each of the following Fourier transforms, use Fourier transform properties (Table 5.1) to determine whether the corresponding time-domain signal is (i) real, imaginary, or neither and (ii) even, odd, or neither. Do this without evaluating the inverse of any of the given transforms.

(a) $X_1(e^{j\omega}) = e^{-j\omega}\sum_{k=1}^{10}(\sin k\omega)$

(b) $X_2(e^{j\omega}) = j\sin(\omega)\cos(5\omega)$

(c) $X_3(e^{j\omega}) = A(\omega) + e^{jB(\omega)}$ where

$$A(\omega) = \begin{cases} 1, & 0 \le |\omega| \le \frac{\pi}{8} \\ 0, & \frac{\pi}{8} < |\omega| \le \pi \end{cases} \quad \text{and } B(\omega) = -\frac{3\omega}{2} + \pi.$$

5.8. Use Tables 5.1 and 5.2 to help determine $x[n]$ when its Fourier transform is

$$X(e^{j\omega}) = \frac{1}{1 - e^{-j\omega}}\left(\frac{\sin\frac{3}{2}\omega}{\sin\frac{\omega}{2}}\right) + 5\pi\delta(\omega), \qquad -\pi < \omega \le \pi$$

5.9. The following four facts are given about a real signal $x[n]$ with Fourier transform $X(e^{jw})$:

1. $x[n] = 0$ for $n > 0$.
2. $x[0] > 0$.
3. $\mathcal{Im}\{X(e^{j\omega})\} = \sin\omega - \sin 2\omega$.
4. $\frac{1}{2\pi}\int_{-\pi}^{\pi}|X(e^{j\omega})|^2\,d\omega = 3$.

Determine $x[n]$.

5.10. Use Tables 5.1 and 5.2 in conjunction with the fact that

$$X(e^{j0}) = \sum_{n=-\infty}^{\infty} x[n]$$

to determine the numerical value of

$$A = \sum_{n=0}^{\infty} n\left(\frac{1}{2}\right)^n.$$

5.11. Consider a signal $g[n]$ with Fourier transform $G(e^{j\omega})$. Suppose

$$g[n] = x_{(2)}[n],$$

where the signal $x[n]$ has a Fourier transform $X(e^{j\omega})$. Determine a real number α such that $0 < \alpha < 2\pi$ and $G(e^{j\omega}) = G(e^{j(\omega-\alpha)})$.

5.12. Let

$$y[n] = \left(\frac{\sin\frac{\pi}{4}n}{\pi n}\right)^2 * \left(\frac{\sin\omega_c n}{\pi n}\right),$$

where $*$ denotes convolution and $|\omega_c| \le \pi$. Determine a stricter constraint on ω_c

which ensures that

$$y[n] = \left(\frac{\sin\frac{\pi}{4}n}{\pi n}\right)^2.$$

5.13. An LTI system with impulse response $h_1[n] = (\frac{1}{3})^n u[n]$ is connected in parallel with another causal LTI system with impulse response $h_2[n]$. The resulting parallel interconnection has the frequency response

$$H(e^{j\omega}) = \frac{-12 + 5e^{-j\omega}}{12 - 7e^{-j\omega} + e^{-j2\omega}}.$$

Determine $h_2[n]$.

5.14. Suppose we are given the following facts about an LTI system S with impulse response $h[n]$ and frequency response $H(e^{j\omega})$:

1. $(\frac{1}{4})^n u[n] \longrightarrow g[n]$, where $g[n] = 0$ for $n \geq 2$ and $n < 0$.
2. $H(e^{j\pi/2}) = 1$.
3. $H(e^{j\omega}) = H(e^{j(\omega-\pi)})$.

Determine $h[n]$.

5.15. Let the inverse Fourier transform of $Y(e^{j\omega})$ be

$$y[n] = \left(\frac{\sin\omega_c n}{\pi n}\right)^2,$$

where $0 < \omega_c < \pi$. Determine the value of ω_c which ensures that

$$Y(e^{j\pi}) = \frac{1}{2}.$$

5.16. The Fourier transform of a particular signal is

$$X(e^{j\omega}) = \sum_{k=0}^{3} \frac{(1/2)^k}{1 - \frac{1}{4}e^{-j(\omega-\pi/2k)}}.$$

It can be shown that

$$x[n] = g[n]q[n],$$

where $g[n]$ is of the form $\alpha^n u[n]$ and $q[n]$ is a periodic signal with period N.

(a) Determine the value of α.
(b) Determine the value of N.
(c) Is $x[n]$ real?

5.17. The signal $x[n] = (-1)^n$ has a fundamental period of 2 and corresponding Fourier series coefficients a_k. Use duality to determine the Fourier series coefficients b_k of the signal $g[n] = a_n$ with a fundamental period of 2.

5.18. Given the fact that

$$a^{|n|} \stackrel{\mathfrak{F}}{\longleftrightarrow} \frac{1 - a^2}{1 - 2a\cos\omega + a^2}, \quad |a| < 1,$$

use duality to determine the Fourier series coefficients of the following continuous-time signal with period $T = 1$:

$$x(t) = \frac{1}{5 - 4\cos(2\pi t)}.$$

5.19. Consider a causal and stable LTI system S whose input $x[n]$ and output $y[n]$ are related through the second-order difference equation

$$y[n] - \frac{1}{6}y[n-1] - \frac{1}{6}y[n-2] = x[n].$$

(a) Determine the frequency response $H(e^{j\omega})$ for the system S.
(b) Determine the impulse response $h[n]$ for the system S.

5.20. A causal and stable LTI system S has the property that

$$\left(\frac{4}{5}\right)^n u[n] \longrightarrow n\left(\frac{4}{5}\right)^n u[n].$$

(a) Determine the frequency response $H(e^{j\omega})$ for the system S.
(b) Determine a difference equation relating any input $x[n]$ and the corresponding output $y[n]$.

BASIC PROBLEMS

5.21. Compute the Fourier transform of each of the following signals:
(a) $x[n] = u[n-2] - u[n-6]$
(b) $x[n] = (\frac{1}{2})^{-n}u[-n-1]$
(c) $x[n] = (\frac{1}{3})^{|n|}u[-n-2]$
(d) $x[n] = 2^n \sin(\frac{\pi}{4}n)u[-n]$
(e) $x[n] = (\frac{1}{2})^{|n|}\cos(\frac{\pi}{8}(n-1))$
(f) $x[n] = \begin{cases} n, & -3 \le n \le 3 \\ 0, & \text{otherwise} \end{cases}$
(g) $x[n] = \sin(\frac{\pi}{2}n) + \cos(n)$
(h) $x[n] = \sin(\frac{5\pi}{3}n) + \cos(\frac{7\pi}{3}n)$
(i) $x[n] = x[n-6]$, and $x[n] = u[n] - u[n-5]$ for $0 \le n \le 5$
(j) $x[n] = (n-1)(\frac{1}{3})^{|n|}$
(k) $x[n] = (\frac{\sin(\pi n/5)}{\pi n})\cos(\frac{7\pi}{2}n)$

5.22. The following are the Fourier transforms of discrete-time signals. Determine the signal corresponding to each transform.
(a) $X(e^{j\omega}) = \begin{cases} 1, & \frac{\pi}{4} \le |\omega| \le \frac{3\pi}{4} \\ 0, & \frac{3\pi}{4} \le |\omega| \le \pi, 0 \le |\omega| < \frac{\pi}{4} \end{cases}$
(b) $X(e^{j\omega}) = 1 + 3e^{-j\omega} + 2e^{-j2\omega} - 4e^{-j3\omega} + e^{-j10\omega}$
(c) $X(e^{j\omega}) = e^{-j\omega/2}$ for $-\pi \le \omega \le \pi$
(d) $X(e^{j\omega}) = \cos^2\omega + \sin^2 3\omega$

(e) $X(e^{j\omega}) = \sum_{k=-\infty}^{\infty}(-1)^k\delta(\omega - \frac{\pi}{2}k)$

(f) $X(e^{j\omega}) = \frac{e^{-j\omega}-\frac{1}{5}}{1-\frac{1}{5}e^{-j\omega}}$

(g) $X(e^{j\omega}) = \frac{1-\frac{1}{3}e^{-j\omega}}{1-\frac{1}{4}e^{-j\omega}-\frac{1}{8}e^{-2j\omega}}$

(h) $X(e^{j\omega}) = \frac{1-(\frac{1}{3})^6 e^{-j6\omega}}{1-\frac{1}{3}e^{-j\omega}}$

5.23. Let $X(e^{j\omega})$ denote the Fourier transform of the signal $x[n]$ depicted in Figure P5.23. Perform the following calculations without explicitly evaluating $X(e^{j\omega})$:

(a) Evaluate $X(e^{j0})$.
(b) Find $\sphericalangle X(e^{j\omega})$.
(c) Evaluate $\int_{-\pi}^{\pi} X(e^{j\omega})d\omega$.
(d) Find $X(e^{j\pi})$.
(e) Determine and sketch the signal whose Fourier transform is $\mathcal{R}e\{x(\omega)\}$.
(f) Evaluate:
  (i) $\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega$
  (ii) $\int_{-\pi}^{\pi} \left|\frac{dX(e^{j\omega})}{d\omega}\right|^2 d\omega$

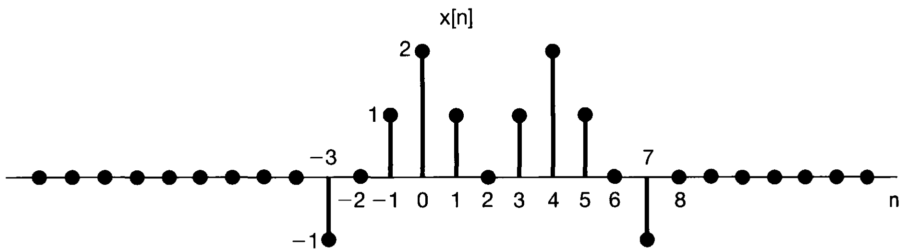


Fig P5.23

5.24. Determine which, if any, of the following signals have Fourier transforms that satisfy each of the following conditions:

1. $\mathcal{R}e\{X(e^{j\omega})\} = 0$.
2. $\mathcal{I}m\{X(e^{j\omega})\} = 0$.
3. There exists a real α such that $e^{j\alpha\omega}X(e^{j\omega})$ is real.
4. $\int_{-\pi}^{\pi} X(e^{j\omega})d\omega = 0$.
5. $X(e^{j\omega})$ periodic.
6. $X(e^{j0}) = 0$.

(a) $x[n]$ as in Figure P5.24(a)
(b) $x[n]$ as in Figure P5.24(b)
(c) $x[n] = (\frac{1}{2})^n u[n]$
(d) $x[n] = (\frac{1}{2})^{|n|}$
(e) $x[n] = \delta[n-1] + \delta[n+2]$
(f) $x[n] = \delta[n-1] + \delta[n+3]$
(g) $x[n]$ as in Figure P5.24(c)
(h) $x[n]$ as in Figure P5.24(d)
(i) $x[n] = \delta[n-1] - \delta[n+1]$

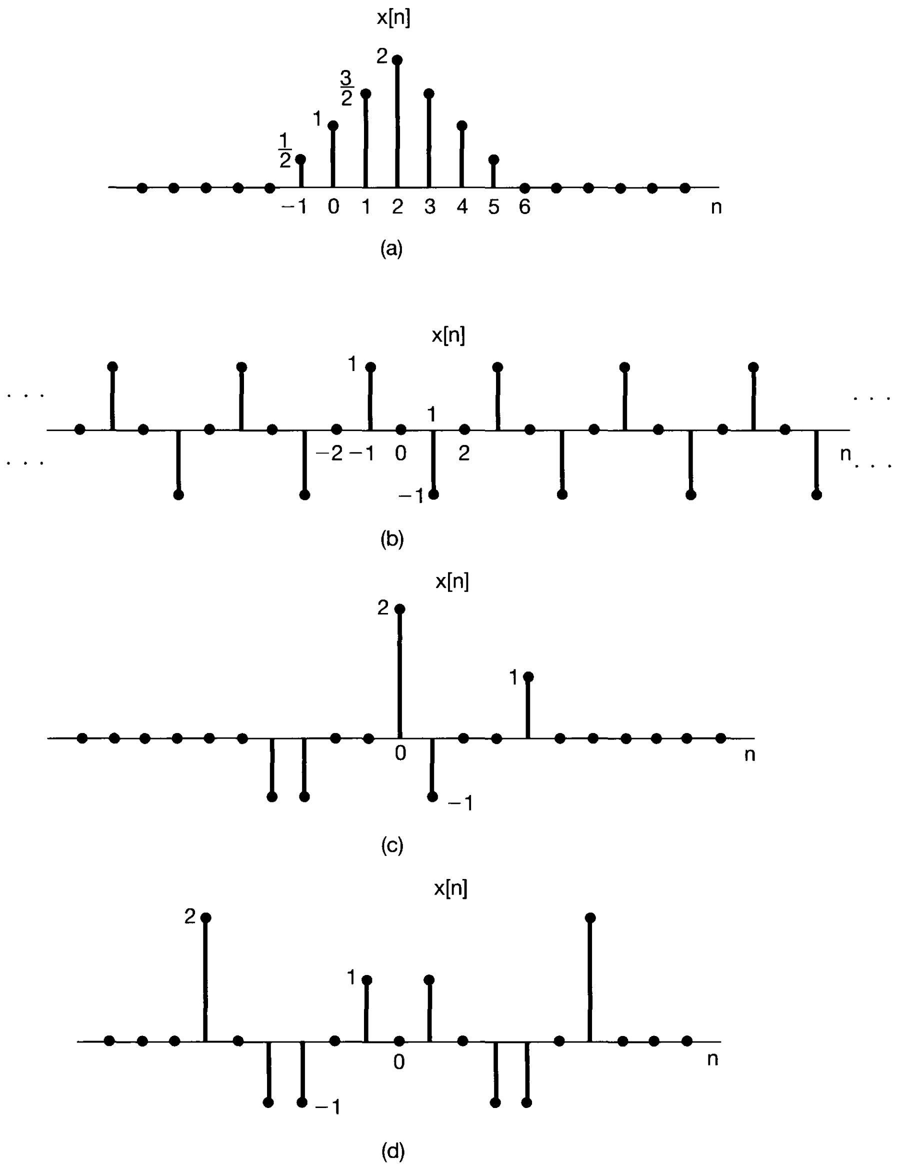


Fig P5.24

5.25. Consider the signal depicted in Figure P5.25. Let the Fourier transform of this signal be written in rectangular form as

$$X(e^{j\omega}) = A(\omega) + jB(\omega).$$

Sketch the function of time corresponding to the transform

$$Y(e^{j\omega}) = [B(\omega) + A(\omega)e^{j\omega}].$$

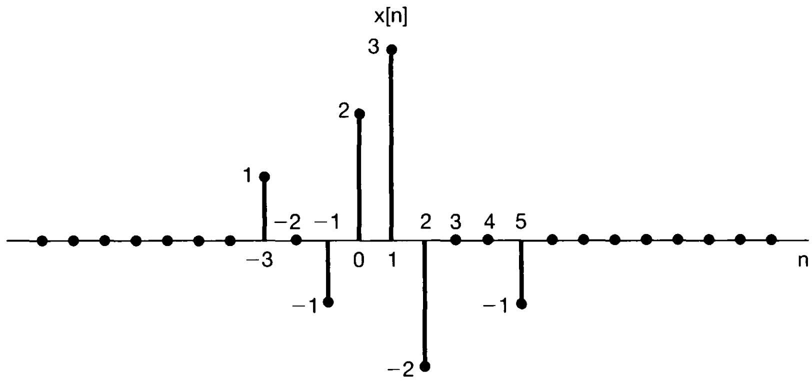


Fig P5.25

5.26. Let $x_1[n]$ be the discrete-time signal whose Fourier transform $X_1(e^{j\omega})$ is depicted in Figure P5.26(a).

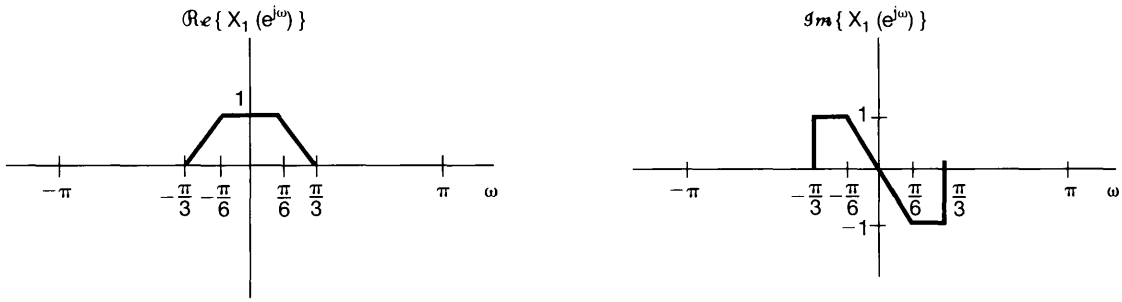
(a) Consider the signal $x_2[n]$ with Fourier transform $X_2(e^{j\omega})$, as illustrated in Figure P5.26(b). Express $x_2[n]$ in terms of $x_1[n]$. [*Hint:* First express $X_2(e^{j\omega})$ in terms of $X_1(e^{j\omega})$, and then use properties of the Fourier transform.]

(b) Repeat part (a) for $x_3[n]$ with Fourier transform $X_3(e^{j\omega})$, as shown in Figure P5.26(c).

(c) Let

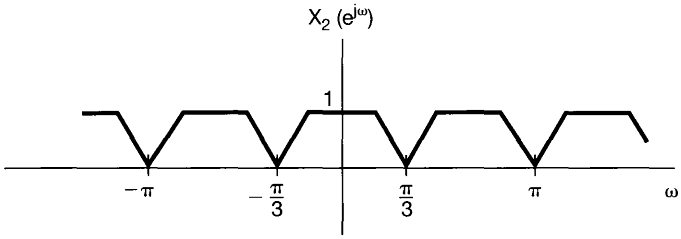
$$\alpha = \frac{\sum_{n=-\infty}^{\infty} n x_1[n]}{\sum_{n=-\infty}^{\infty} x_1[n]}.$$

This quantity, which is the center of gravity of the signal $x_1[n]$, is usually referred to as the *delay time* of $x_1[n]$. Find α. (You can do this without first determining $x_1[n]$ explicitly.)

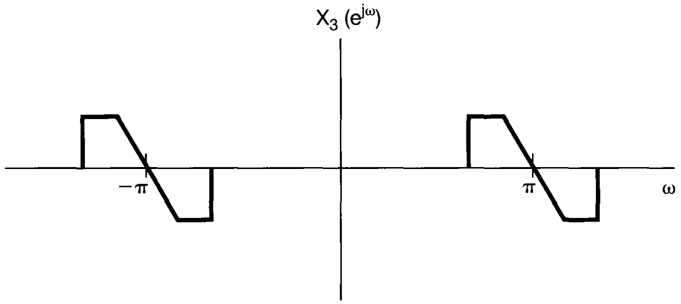


(a)

Fig P5.26a



(b)



(c)

Fig P5.26b,c

(d) Consider the signal $x_4[n] = x_1[n] * h[n]$, where

$$h[n] = \frac{\sin(\pi n/6)}{\pi n}.$$

Sketch $X_4(e^{j\omega})$.

5.27. **(a)** Let $x[n]$ be a discrete-time signal with Fourier transform $X(e^{j\omega})$, which is illustrated in Figure P5.27. Sketch the Fourier transform of

$$w[n] = x[n]p[n]$$

for each of the following signals $p[n]$:

(i) $p[n] = \cos \pi n$

(ii) $p[n] = \cos(\pi n/2)$

(iii) $p[n] = \sin(\pi n/2)$

(iv) $p[n] = \sum_{k=-\infty}^{\infty} \delta[n - 2k]$

(v) $p[n] = \sum_{k=-\infty}^{\infty} \delta[n - 4k]$

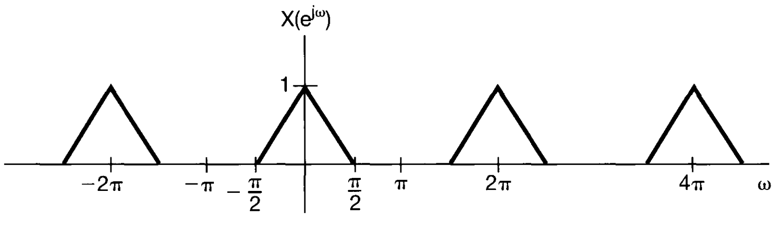


Fig P5.27

(b) Suppose that the signal $w[n]$ of part (a) is applied as the input to an LTI system with unit sample response

$$h[n] = \frac{\sin(\pi n/2)}{\pi n}.$$

Determine the output $y[n]$ for each of the choices of $p[n]$ in part (a).

5.28. The signals $x[n]$ and $g[n]$ are known to have Fourier transforms $X(e^{j\omega})$ and $G(e^{j\omega})$, respectively. Furthermore, $X(e^{j\omega})$ and $G(e^{j\omega})$ are related as follows:

$$\frac{1}{2\pi}\int_{-\pi}^{+\pi} X(e^{j\theta})G(e^{j(\omega-\theta)})d\theta = 1 + e^{-j\omega} \qquad \text{(P5.28–1)}$$

(a) If $x[n] = (-1)^n$, determine a sequence $g[n]$ such that its Fourier transform $G(e^{j\omega})$ satisfies eq. (P5.28–1). Are there other possible solutions for $g[n]$?

(b) Repeat the previous part for $x[n] = (\frac{1}{2})^n u[n]$.

5.29. **(a)** Consider a discrete-time LTI system with impulse response

$$h[n] = \left(\frac{1}{2}\right)^n u[n].$$

Use Fourier transforms to determine the response to each of the following input signals:

(i) $x[n] = (\frac{3}{4})^n u[n]$

(ii) $x[n] = (n+1)(\frac{1}{4})^n u[n]$

(iii) $x[n] = (-1)^n$

(b) Suppose that

$$h[n] = \left[\left(\frac{1}{2}\right)^n \cos\left(\frac{\pi n}{2}\right)\right] u[n].$$

Use Fourier transforms to determine the response to each of the following inputs:

(i) $x[n] = (\frac{1}{2})^n u[n]$

(ii) $x[n] = \cos(\pi n/2)$

(c) Let $x[n]$ and $h[n]$ be signals with the following Fourier transforms:

$$X(e^{j\omega}) = 3e^{j\omega} + 1 - e^{-j\omega} + 2e^{-j3\omega},$$
$$H(e^{j\omega}) = -e^{j\omega} + 2e^{-2j\omega} + e^{j4\omega}.$$

Determine $y[n] = x[n] * h[n]$.

5.30. In Chapter 4, we indicated that the continuous-time LTI system with impulse response

$$h(t) = \frac{W}{\pi}\operatorname{sinc}\left(\frac{Wt}{\pi}\right) = \frac{\sin Wt}{\pi t}$$

plays a very important role in LTI system analysis. The same is true of the discrete-time LTI system with impulse response

$$h[n] = \frac{W}{\pi}\operatorname{sinc}\left(\frac{Wn}{\pi}\right) = \frac{\sin Wn}{\pi n}.$$

(a) Determine and sketch the frequency response for the system with impulse response $h[n]$.

(b) Consider the signal

$$x[n] = \sin\left(\frac{\pi n}{8}\right) - 2\cos\left(\frac{\pi n}{4}\right).$$

Suppose that this signal is the input to LTI systems with the following impulse responses. Determine the output in each case.

(i) $h[n] = \frac{\sin(\pi n/6)}{\pi n}$

(ii) $h[n] = \frac{\sin(\pi n/6)}{\pi n} + \frac{\sin(\pi n/2)}{\pi n}$

(iii) $h[n] = \frac{\sin(\pi n/6)\sin(\pi n/3)}{\pi^2 n^2}$

(iv) $h[n] = \frac{\sin(\pi n/6)\sin(\pi n/3)}{\pi n}$

(c) Consider an LTI system with unit sample response

$$h[n] = \frac{\sin(\pi n/3)}{\pi n}.$$

Determine the output for each of the following inputs:

(i) $x[n] =$ the square wave depicted in Figure P5.30

(ii) $x[n] = \sum_{k=-\infty}^{\infty} \delta[n - 8k]$

(iii) $x[n] = (-1)^n$ times the square wave depicted in Figure P5.30

(iv) $x[n] = \delta[n + 1] + \delta[n - 1]$

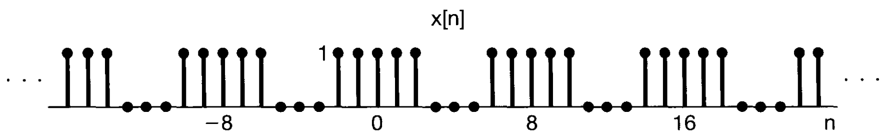


Fig P5.30

5.31. An LTI system S with impulse response $h[n]$ and frequency response $H(e^{j\omega})$ is known to have the property that, when $-\pi \le \omega_0 \le \pi$,

$$\cos \omega_0 n \longrightarrow \omega_0 \cos \omega_0 n.$$

(a) Determine $H(e^{j\omega})$.

(b) Determine $h[n]$.

5.32. Let $h_1[n]$ and $h_2[n]$ be the impulse responses of causal LTI systems, and let $H_1(e^{j\omega})$ and $H_2(e^{j\omega})$ be the corresponding frequency responses. Under these conditions, is the following equation true in general or not? Justify your answer.

$$\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} H_1(e^{j\omega})d\omega\right]\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} H_2(e^{j\omega})d\omega\right] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_1(e^{j\omega})H_2(e^{j\omega})d\omega.$$

5.33. Consider a causal LTI system described by the difference equation

$$y[n] + \frac{1}{2}y[n-1] = x[n].$$

(a) Determine the frequency response $H(e^{j\omega})$ of this system.
(b) What is the response of the system to the following inputs?
  (i) $x[n] = (\frac{1}{2})^n u[n]$
  (ii) $x[n] = (-\frac{1}{2})^n u[n]$
  (iii) $x[n] = \delta[n] + \frac{1}{2}\delta[n-1]$
  (iv) $x[n] = \delta[n] - \frac{1}{2}\delta[n-1]$
(c) Find the response to the inputs with the following Fourier transforms:
  (i) $X(e^{j\omega}) = \frac{1-\frac{1}{4}e^{-j\omega}}{1+\frac{1}{2}e^{-j\omega}}$
  (ii) $X(e^{j\omega}) = \frac{1+\frac{1}{2}e^{-j\omega}}{1-\frac{1}{4}e^{-j\omega}}$
  (iii) $X(e^{j\omega}) = \frac{1}{(1-\frac{1}{4}e^{-j\omega})(1+\frac{1}{2}e^{-j\omega})}$
  (iv) $X(e^{j\omega}) = 1 + 2e^{-3j\omega}$

5.34. Consider a system consisting of the cascade of two LTI systems with frequency responses

$$H_1(e^{j\omega}) = \frac{2 - e^{-j\omega}}{1 + \frac{1}{2}e^{-j\omega}}$$

and

$$H_2(e^{j\omega}) = \frac{1}{1 - \frac{1}{2}e^{-j\omega} + \frac{1}{4}e^{-j2\omega}}.$$

(a) Find the difference equation describing the overall system.
(b) Determine the impulse response of the overall system.

5.35. A causal LTI system is described by the difference equation

$$y[n] - ay[n-1] = bx[n] + x[n-1],$$

where a is real and less than 1 in magnitude.
(a) Find a value of b such that the frequency response of the system satisfies

$$|H(e^{j\omega})| = 1, \text{ for all } \omega.$$

This kind of system is called an *all-pass system*, as it does not attenuate the input $e^{j\omega n}$ for *any* value of ω. Use the value of b that you have found in the rest of the problem.
(b) Roughly sketch $\sphericalangle H(e^{j\omega})$, $0 \le \omega \le \pi$, when $a = \frac{1}{2}$.
(c) Roughly sketch $\sphericalangle H(e^{j\omega})$, $0 \le \omega \le \pi$, when $a = -\frac{1}{2}$.

(d) Find and plot the output of this system with $a = -\frac{1}{2}$ when the input is

$$x[n] = \left(\frac{1}{2}\right)^n u[n].$$

From this example, we see that a nonlinear change in phase can have a significantly different effect on a signal than the time shift that results from a linear phase.

5.36. **(a)** Let $h[n]$ and $g[n]$ be the impulse responses of two stable discrete-time LTI systems that are inverses of each other. What is the relationship between the frequency responses of these two systems?

(b) Consider causal LTI systems described by the following difference equations. In each case, determine the impulse response of the inverse system and the difference equation that characterizes the inverse.

(i) $y[n] = x[n] - \frac{1}{4}x[n-1]$

(ii) $y[n] + \frac{1}{2}y[n-1] = x[n]$

(iii) $y[n] + \frac{1}{2}y[n-1] = x[n] - \frac{1}{4}x[n-1]$

(iv) $y[n] + \frac{5}{4}y[n-1] - \frac{1}{8}y[n-2] = x[n] - \frac{1}{4}x[n-1] - \frac{1}{8}x[n-2]$

(v) $y[n] + \frac{5}{4}y[n-1] - \frac{1}{8}y[n-2] = x[n] - \frac{1}{2}x[n-1]$

(vi) $y[n] + \frac{5}{4}y[n-1] - \frac{1}{8}y[n-2] = x[n]$

(c) Consider the causal, discrete-time LTI system described by the difference equation

$$y[n] + y[n-1] + \frac{1}{4}y[n-2] = x[n-1] - \frac{1}{2}x[n-2]. \qquad \text{(P5.36–1)}$$

What is the inverse of this system? Show that the inverse is not causal. Find another causal LTI system that is an "inverse with delay" of the system described by eq. (P5.36–1). Specifically, find a causal LTI system such that the output $w[n]$ in Figure P5.36 equals $x[n-1]$.

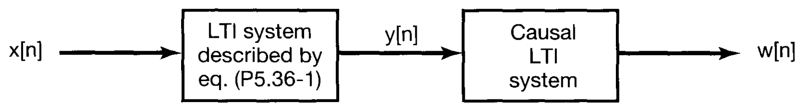


Fig P5.36

ADVANCED PROBLEMS

5.37. Let $X(e^{j\omega})$ be the Fourier transform of $x[n]$. Derive expressions in terms of $X(e^{j\omega})$ for the Fourier transforms of the following signals. (Do not assume that $x[n]$ is real.)

(a) $\mathcal{R}e\{x[n]\}$

(b) $x^*[-n]$

(c) $\mathcal{E}v\{x[n]\}$

5.38. Let $X(e^{j\omega})$ be the Fourier transform of a real signal $x[n]$. Show that $x[n]$ can be written as

$$x[n] = \int_0^{\pi} \{B(\omega)\cos\omega + C(\omega)\sin\omega\}d\omega$$

by finding expressions for $B(\omega)$ and $C(\omega)$ in terms of $X(e^{j\omega})$.

5.39. Derive the convolution property

$$x[n] * h[n] \stackrel{\mathcal{F}}{\longleftrightarrow} X(e^{j\omega})H(e^{j\omega}).$$

5.40. Let $x[n]$ and $h[n]$ be two signals, and let $y[n] = x[n] * h[n]$. Write two expressions for $y[0]$, one (using the convolution sum directly) in terms of $x[n]$ and $h[n]$, and one (using the convolution property of Fourier transforms) in terms of $X(e^{j\omega})$ and $H(e^{j\omega})$. Then, by a judicious choice of $h[n]$, use these two expressions to derive Parseval's relation—that is,

$$\sum_{n=-\infty}^{+\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega.$$

In a similar fashion, derive the following generalization of Parseval's relation:

$$\sum_{n-\infty}^{+\infty} x[n]z^*[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})Z^*(e^{j\omega})d\omega.$$

5.41 Let $\tilde{x}[n]$ be a periodic signal with period N. A finite-duration signal $x[n]$ is related to $\tilde{x}[n]$ through

$$x[n] = \begin{cases} \tilde{x}[n], & n_0 \le n \le n_0 + N - 1 \\ 0, & \text{otherwise} \end{cases},$$

for some integer n_0. That is, $x[n]$ is equal to $\tilde{x}[n]$ over one period and zero elsewhere.

(a) If $\tilde{x}[n]$ has Fourier series coefficients a_k and $x[n]$ has Fourier transform $X(e^{j\omega})$, show that

$$a_k = \frac{1}{N}X(e^{j2\pi k/N})$$

regardless of the value of n_0.

(b) Consider the following two signals:

$$x[n] = u[n] - u[n-5]$$

$$\tilde{x}[n] = \sum_{k=-\infty}^{\infty} x[n-kN]$$

where N is a positive integer. Let a_k denote the Fourier coefficients of $\tilde{x}[n]$ and let $X(e^{j\omega})$ denote the Fourier transform of $x[n]$.

(i) Determine a closed-form expression for $X(e^{j\omega})$.

(ii) Using the result of part (i), determine an expression for the Fourier coefficients a_k.

5.42. In this problem, we derive the frequency-shift property of the discrete-time Fourier transform as a special case of the multiplication property. Let $x[n]$ be any discrete-time signal with Fourier transform $X(e^{j\omega})$, and let

$$g[n] = e^{j\omega_0 n} x[n].$$

(a) Determine and sketch the Fourier transform of

$$p[n] = e^{j\omega_0 n}.$$

(b) The multiplication property of the Fourier transform tells us that, since

$$g[n] = p[n]x[n],$$

$$G(e^{j\omega}) = \frac{1}{2\pi} \int_{<2\pi>} X(e^{j\theta}) P(e^{j(\omega-\theta)}) d\theta.$$

Evaluate this integral to show that

$$G(e^{j\omega}) = X(e^{j(\omega-\omega_0)}).$$

5.43. Let $x[n]$ be a signal with Fourier transform $X(e^{j\omega})$, and let

$$g[n] = x[2n]$$

be a signal whose Fourier transform is $G(e^{j\omega})$. In this problem, we derive the relationship between $G(e^{j\omega})$ and $X(e^{j\omega})$.

(a) Let

$$v[n] = \frac{(e^{-j\pi n} x[n]) + x[n]}{2}.$$

Express the Fourier transform $V(e^{j\omega})$ of $v[n]$ in terms of $X(e^{j\omega})$.

(b) Noting that $v[n] = 0$ for n odd, show that the Fourier transform of $v[2n]$ is equal to $V(e^{j\frac{\omega}{2}})$.

(c) Show that

$$x[2n] = v[2n].$$

It follows that

$$G(e^{j\omega}) = V(e^{j\omega/2}).$$

Now use the result of part (a) to express $G(e^{j\omega})$ in terms of $X(e^{j\omega})$.

5.44. **(a)** Let

$$x_1[n] = \cos\left(\frac{\pi n}{3}\right) + \sin\left(\frac{\pi n}{2}\right)$$

be a signal, and let $X_1(e^{j\omega})$ denote the Fourier transform of $x_1[n]$. Sketch $x_1[n]$, together with the signals with the following Fourier transforms:

(i) $X_2(e^{j\omega}) = X_1(e^{j\omega})e^{j\omega}, \ |\omega| < \pi$

(ii) $X_3(e^{j\omega}) = X_1(e^{j\omega})e^{-j3\omega/2}, \ |\omega| < \pi$

(b) Let

$$w(t) = \cos\left(\frac{\pi t}{3T}\right) + \sin\left(\frac{\pi t}{2T}\right)$$

be a continuous-time signal. Note that $x_1[n]$ can be regarded as a sequence of evenly spaced samples of $w(t)$; that is,

$$x_1[n] = w(nT).$$

Show that

$$x_2[n] = w(nT - \alpha)$$

and

$$x_3[n] = w(nT - \beta)$$

and specify the values of α and β. From this result we can conclude that $x_2[n]$ and $x_3[n]$ are also evenly spaced samples of $w(t)$.

5.45. Consider a discrete-time signal $x[n]$ with Fourier transform as illustrated in Figure P5.45. Provide dimensioned sketches of the following continuous-time signals:

(a) $x_1(t) = \sum_{n=-\infty}^{\infty} x[n]e^{j(2\pi/10)nt}$

(b) $x_2(t) = \sum_{n=-\infty}^{\infty} x[-n]e^{j(2\pi/10)nt}$

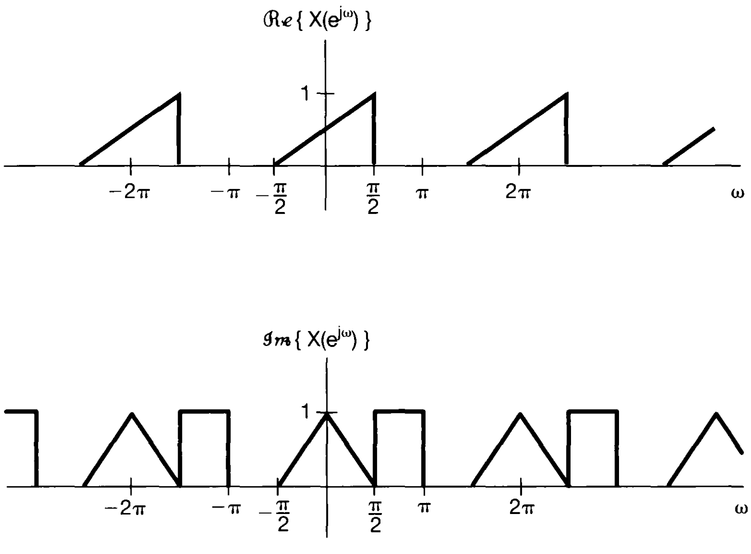


Fig P5.45

(c) $x_3(t) = \sum_{n=-\infty}^{\infty} \mathcal{O}d\{x[n]\}e^{j(2\pi/8)nt}$

(d) $x_4(t) = \sum_{n=-\infty}^{\infty} \mathcal{R}e\{x[n]\}e^{j(2\pi/6)nt}$

5.46. In Example 5.1, we showed that for $|\alpha| < 1$,

$$\alpha^n u[n] \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{1 - \alpha e^{-j\omega}}.$$

(a) Use properties of the Fourier transform to show that

$$(n+1)\alpha^n u[n] \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{(1 - \alpha e^{-j\omega})^2}.$$

(b) Show by induction that the inverse Fourier transform of

$$X(e^{j\omega}) = \frac{1}{(1 - \alpha e^{-j\omega})^r}$$

is

$$x[n] = \frac{(n+r-1)!}{n!(r-1)!}\alpha^n u[n].$$

5.47. Determine whether each of the following statements is true or false. Justify your answers. In each statement, the Fourier transform of $x[n]$ is denoted by $X(e^{j\omega})$.

(a) If $X(e^{j\omega}) = X(e^{j(\omega-1)})$, then $x[n] = 0$ for $|n| > 0$.

(b) If $X(e^{j\omega}) = X(e^{j(\omega-\pi)})$, then $x[n] = 0$ for $|n| > 0$.

(c) If $X(e^{j\omega}) = X(e^{j\omega/2})$, then $x[n] = 0$ for $|n| > 0$.

(d) If $X(e^{j\omega}) = X(e^{j2\omega})$, then $x[n] = 0$ for $|n| > 0$.

5.48. We are given a discrete-time, linear, time-invariant, causal system with input denoted by $x[n]$ and output denoted by $y[n]$. This system is specified by the following *pair* of difference equations, involving an intermediate signal $w[n]$:

$$y[n] + \frac{1}{4}y[n-1] + w[n] + \frac{1}{2}w[n-1] = \frac{2}{3}x[n],$$

$$y[n] - \frac{5}{4}y[n-1] + 2w[n] - 2w[n-1] = -\frac{5}{3}x[n].$$

(a) Find the frequency response and unit sample response of the system.

(b) Find a single difference equation relating $x[n]$ and $y[n]$ for the system.

5.49. **(a)** A particular discrete-time system has input $x[n]$ and output $y[n]$. The Fourier transforms of these signals are related by the equation

$$Y(e^{j\omega}) = 2X(e^{j\omega}) + e^{-j\omega}X(e^{j\omega}) - \frac{dX(e^{j\omega})}{d\omega}.$$

(i) Is the system linear? Clearly justify your answer.

(ii) Is the system time invariant? Clearly justify your answer.

(iii) What is $y[n]$ if $x[n] = \delta[n]$?

(b) Consider a discrete-time system for which the transform $Y(e^{j\omega})$ of the output is related to the transform of the input through the relation

$$Y(e^{j\omega}) = \int_{\omega-\pi/4}^{\omega+\pi/4} X(e^{j\omega})d\omega.$$

Find an expression for $y[n]$ in terms of $x[n]$.

5.50. **(a)** Suppose we want to design a discrete-time LTI system which has the property that if the input is

$$x[n] = \left(\frac{1}{2}\right)^n u[n] - \frac{1}{4}\left(\frac{1}{2}\right)^{n-1} u[n-1],$$

then the output is

$$y[n] = \left(\frac{1}{3}\right)^n u[n].$$

(i) Find the impulse response *and* frequency response of a discrete-time LTI system that has the foregoing property.

(ii) Find a difference equation relating $x[n]$ and $y[n]$ that characterizes the system.

(b) Suppose that a system has the response $(1/4)^n u[n]$ to the input $(n+2)(1/2)^n u[n]$. If the output of this system is $\delta[n] - (-1/2)^n u[n]$, what is the input?

5.51. **(a)** Consider a discrete-time system with unit sample response

$$h[n] = \left(\frac{1}{2}\right)^n u[n] + \frac{1}{2}\left(\frac{1}{4}\right)^n u[n].$$

Determine a linear constant-coefficient difference equation relating the input and output of the system.

(b) Figure P5.51 depicts a block diagram implementation of a causal LTI system.

(i) Find a difference equation relating $x[n]$ and $y[n]$ for this system.

(ii) What is the frequency response of the system?

(iii) Determine the system's impulse response.

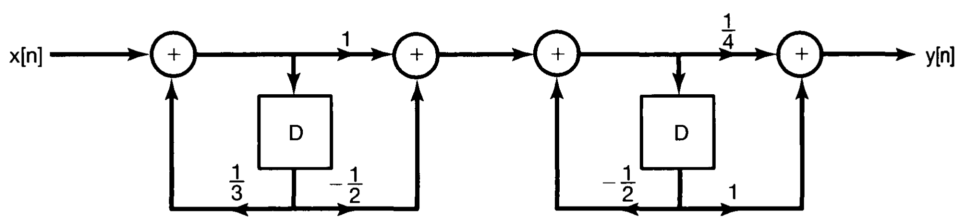


Fig P5.51

5.52. **(a)** Let $h[n]$ be the impulse response of a real, causal, discrete-time LTI system. Show that the system is completely specified by the real part of its frequency response. (*Hint:* Show how $h[n]$ can be recovered from $\mathcal{E}v\{h[n]\}$. What is the Fourier transform of $\mathcal{E}v\{h[n]\}$?) This is the discrete-time counterpart of the *real-part sufficiency* property of causal LTI systems considered in Problem 4.47 for continuous-time systems.

(b) Let $h[n]$ be real and causal. If

$$\mathcal{R}e\{H(e^{j\omega})\} = 1 + \alpha \cos 2\omega (\alpha \text{ real}),$$

determine $h[n]$ and $H(e^{j\omega})$.

(c) Show that $h[n]$ can be completely recovered from knowledge of $\mathcal{I}m\{H(e^{j\omega})\}$ and $h[0]$.

(d) Find two real, causal LTI systems whose frequency responses have imaginary parts equal to $\sin \omega$.

EXTENSION PROBLEMS

5.53. One of the reasons for the tremendous growth in the use of discrete-time methods for the analysis and synthesis of signals and systems was the development of exceedingly efficient tools for performing Fourier analysis of discrete-time sequences. At the heart of these methods is a technique that is very closely allied with discrete-time Fourier analysis and that is ideally suited for use on a digital computer or for implementation in digital hardware. This technique is the *discrete Fourier transform (DFT)* for finite-duration signals.

Let $x[n]$ be a signal of finite duration; that is, there is an integer N_1 so that

$$x[n] = 0, \qquad \text{outside the interval } 0 \leq n \leq N_1 - 1$$

Furthermore, let $X(e^{j\omega})$ denote the Fourier transform of $x[n]$. We can construct a periodic signal $\tilde{x}[n]$ that is equal to $x[n]$ over one period. Specifically, let $N \geq N_1$ be a given integer, and let $\tilde{x}[n]$ be periodic with period N and such that

$$\tilde{x}[n] = x[n], \qquad 0 \leq n \leq N - 1$$

The Fourier series coefficients for $\tilde{x}[n]$ are given by

$$a_k = \frac{1}{N} \sum_{\langle N \rangle} \tilde{x}[n] e^{-jk(2\pi/N)n}$$

Choosing the interval of summation to be that over which $\tilde{x}[n] = x[n]$, we obtain

$$a_k = \frac{1}{N} \sum_{n=0}^{N-1} x[n] e^{-jk(2\pi/N)n} \qquad \text{(P5.53–1)}$$

The set of coefficients defined by eq. (P5.53–1) comprise the DFT of $x[n]$. Specifically, the DFT of $x[n]$ is usually denoted by $\tilde{X}[k]$, and is defined as

$$\tilde{X}[k] = a_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-jk(2\pi/N)n}, \qquad k = 0, 1, \ldots, N-1 \qquad \text{(P5.53–2)}$$

The importance of the DFT stems from several facts. First note that the original finite duration signal can be recovered from its DFT. Specifically, we have

$$x[n] = \sum_{k=0}^{N-1} \tilde{X}[k]e^{jk(2\pi/N)n}, \qquad n = 0, 1, \ldots, N-1 \qquad \text{(P5.53–3)}$$

Thus, the finite-duration signal can either be thought of as being specified by the finite set of nonzero values it assumes or by the finite set of values of $\tilde{X}[k]$ in its DFT. A second important feature of the DFT is that there is an extremely fast algorithm, called the *fast Fourier transform (FFT),* for its calculation (see Problem 5.54 for an introduction to this extremely important technique). Also, because of its close relationship to the discrete-time Fourier series and transform, the DFT inherits some of their important properties.

(a) Assume that $N \geq N_1$. Show that

$$\tilde{X}[k] = \frac{1}{N}X\left(e^{j(2\pi k/N)}\right)$$

where $\tilde{X}[k]$ is the DFT of $x[n]$. That is, the DFT corresponds to samples of $X(e^{j\omega})$ taken every $2\pi/N$. Equation (P5.53–3) leads us to conclude that $x[n]$ can be uniquely represented by these samples of $X(e^{j\omega})$.

(b) Let us consider samples of $X(e^{j\omega})$ taken every $2\pi/M$, where $M < N_1$. These samples correspond to more than one sequence of duration N_1. To illustrate this, consider the two signals $x_1[n]$ and $x_2[n]$ depicted in Figure P5.53. Show that if we choose $M = 4$, we have

$$X_1\left(e^{j(2\pi k/4)}\right) = X_2\left(e^{j(2\pi k/4)}\right)$$

for all values of k.

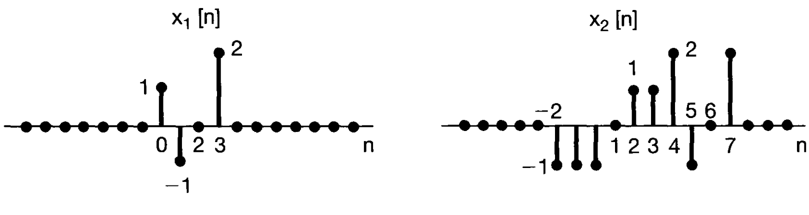


Fig P5.53

5.54. As indicated in Problem 5.53, there are many problems of practical importance in which one wishes to calculate the discrete Fourier transform (DFT) of discrete-time signals. Often, these signals are of quite long duration, and in such cases it is very important to use computationally efficient procedures. One of the reasons for the significant increase in the use of computerized techniques for the analysis of signals was the development of a very efficient technique known as the fast Fourier transform (FFT) algorithm for the calculation of the DFT of finite-duration sequences. In this problem, we develop the principle on which the FFT is based.

Let $x[n]$ be a signal that is 0 outside the interval $0 \leq n \leq N_1 - 1$. For $N \geq N_1$, the N-point DFT of $x[n]$ is given by

$$\tilde{X}[k] = \frac{1}{N}\sum_{k=0}^{N-1} x[n]e^{-jk(2\pi/N)n}, \quad k = 0, 1, \ldots, N-1. \qquad \text{(P5.54–1)}$$

It is convenient to write eq. (P5.54–1) as

$$\tilde{X}[k] = \frac{1}{N}\sum_{k=0}^{N-1} x[n]W_N^{nk}, \qquad \text{(P5.54–2)}$$

where

$$W_N = e^{-j2\pi/N}.$$

(a) One method for calculating $\tilde{X}[k]$ is by direct evaluation of eq. (P5.54–2). A useful measure of the complexity of such a computation is the total number of complex multiplications required. Show that the number of complex multiplications required to evaluate eq. (P5.54–2) directly, for $k = 0, 1, \ldots, N-1$, is N^2. Assume that $x[n]$ is complex and that the required values of W_N^{nk} have been precomputed and stored in a table. For simplicity, *do not* exploit the fact that, for certain values of n and k, W_N^{nk} is equal to ± 1 or $\pm j$ and hence does not, strictly speaking, require a full complex multiplication.

(b) Suppose that N is even. Let $f[n] = x[2n]$ represent the even-indexed samples of $x[n]$, and let $g[n] = x[2n+1]$ represent the odd-indexed samples.

(i) Show that $f[n]$ and $g[n]$ are zero outside the interval $0 \leq n \leq (N/2) - 1$.

(ii) Show that the N-point DFT $\tilde{X}[k]$ of $x[n]$ can be expressed as

$$\begin{aligned}\tilde{X}[k] &= \frac{1}{N}\sum_{n=0}^{(N/2)-1} f[n]W_{N/2}^{nk} + \frac{1}{N}W_N^k\sum_{n=0}^{(N/2)-1} g[n]W_{N/2}^{nk} \\ &= \frac{1}{2}\tilde{F}[k] + \frac{1}{2}W_N^k\tilde{G}[k], \quad k = 0, 1, \ldots, N-1, \qquad \text{(P5.54–3)}\end{aligned}$$

where

$$\tilde{F}[k] = \frac{2}{N}\sum_{n=0}^{(N/2)-1} f[n]W_{N/2}^{nk},$$

$$\tilde{G}[k] = \frac{2}{N}\sum_{n=0}^{(N/2)-1} g[n]W_{N/2}^{nk}.$$

(iii) Show that, for all k,

$$\tilde{F}\left[k + \frac{N}{2}\right] = \tilde{F}[k],$$

$$\tilde{G}\left[k + \frac{N}{2}\right] = \tilde{G}[k].$$

Note that $\tilde{F}[k]$, $k = 0, 1, \ldots, (N/2) - 1$, and $\tilde{G}[k]$, $k = 0, 1, \ldots, (N/2) - 1$, are the $(N/2)$-point DFTs of $f[n]$ and $g[n]$, respectively. Thus, eq. (P5.54–3) indicates that the length-N DFT of $x[n]$ can be calculated in terms of two DFTs of length $N/2$.

(iv) Determine the number of complex multiplications required to compute $\tilde{X}[k]$, $k = 0, 1, 2, \ldots, N - 1$, from eq. (P5.54–3) by first computing $\tilde{F}[k]$ and $\tilde{G}[k]$. [Make the same assumptions about multiplications as in part (a), and ignore the multiplications by the quantity 1/2 in eq. (P5.54–3).]

(c) If, like N, $N/2$ is even, then $f[n]$ and $g[n]$ can each be decomposed into sequences of even- and odd-indexed samples, and therefore, their DFTs can be computed using the same process as in eq. (P5.54–3). Furthermore, if N is an integer power of 2, we can continue to iterate the process, thus achieving significant savings in computation time. With this procedure, approximately how many complex multiplications are required for $N = 32, 256, 1{,}024$, and 4,096? Compare this to the direct method of calculation in part (a).

5.55. In this problem we introduce the concept of *windowing*, which is of great importance both in the design of LTI systems and in the spectral analysis of signals. Windowing is the operation of taking a signal $x[n]$ and multiplying it by a finite-duration *window signal* $w[n]$. That is,

$$p[n] = x[n]w[n].$$

Note that $p[n]$ is also of finite duration.

The importance of windowing in spectral analysis stems from the fact that in numerous applications one wishes to compute the Fourier transform of a signal that has been measured. Since in practice we can measure a signal $x[n]$ only over a finite time interval (the *time window*), the actual signal available for spectral analysis is

$$p[n] = \begin{cases} x[n], & -M \le n \le M \\ 0, & \text{otherwise} \end{cases},$$

where $-M \le n \le M$ is the time window. Thus,

$$p[n] = x[n]w[n],$$

where $w[n]$ is the *rectangular window;* that is,

$$w[n] = \begin{cases} 1, & -M \le n \le M \\ 0, & \text{otherwise} \end{cases}. \qquad \text{(P5.55–1)}$$

Windowing also plays a role in LTI system design. Specifically, for a variety of reasons (such as the potential utility of the FFT algorithm; see Problem P5.54), it is often advantageous to design a system that has an impulse response of finite duration to achieve some desired signal-processing objective. That is, we often begin with a desired frequency response $H(e^{j\omega})$ whose inverse transform $h[n]$ is an impulse response of infinite (or at least excessively long) duration. What is required then is the construction of an impulse response $g[n]$ of finite duration whose transform $G(e^{j\omega})$ adequately approximates $H(e^{j\omega})$. One general approach to choosing $g[n]$ is to find a window function $w[n]$ such that the transform of $h[n]w[n]$ meets the desired specifications for $G(e^{j\omega})$.

Clearly, the windowing of a signal has an effect on the resulting spectrum. In this problem, we illustrate that effect.

(a) To gain some understanding of the effect of windowing, consider windowing the signal

$$x[n] = \sum_{k=-\infty}^{\infty} \delta[n-k]$$

using the rectangular window signal given in eq. (P5.55–1).

(i) What is $X(e^{j\omega})$?

(ii) Sketch the transform of $p[n] = x[n]w[n]$ when $M = 1$.

(iii) Do the same for $M = 10$.

(b) Next, consider a signal $x[n]$ whose Fourier transform is specified by

$$X(e^{j\omega}) = \begin{cases} 1, & |\omega| < \pi/4 \\ 0, & \pi/4 < |\omega| \leq \pi \end{cases}.$$

Let $p[n] = x[n]w[n]$, where $w[n]$ is the rectangular window of eq. (P5.55–1). Roughly sketch $P(e^{j\omega})$ for $M = 4$, 8, and 16.

(c) One of the problems with the use of a rectangular window is that it introduces ripples in the transform $P(e^{j\omega})$. (This is in fact directly related to the Gibbs phenomenon.) For that reason, a variety of other window signals have been developed. These signals are tapered; that is, they go from 0 to 1 more gradually than the abrupt transition of the rectangular window. The result is a reduction in the *amplitude* of the ripples in $P(e^{j\omega})$ at the expense of adding a bit of distortion in terms of further smoothing of $X(e^{j\omega})$.

To illustrate the points just made, consider the signal $x[n]$ described in part (b), and let $p[n] = x[n]w[n]$, where $w[n]$ is the *triangular* or *Bartlett window;* that is,

$$w[n] = \begin{cases} 1 - \frac{|n|}{M+1}, & -M \leq n \leq M \\ 0, & \text{otherwise} \end{cases}.$$

Roughly sketch the Fourier transform of $p[n] = x[n]w[n]$ for $M = 4$, 8, and 16. [*Hint:* Note that the triangular signal can be obtained as a convolution of a rectangular signal with itself. This fact leads to a convenient expression for $W(e^{j\omega})$.]

(d) Let $p[n] = x[n]w[n]$, where $w[n]$ is a raised cosine signal known as the *Hanning window;* i.e.,

$$w[n] = \begin{cases} \frac{1}{2}[1 + \cos(\pi n/M)], & -M \le n \le M \\ 0, & \text{otherwise} \end{cases}.$$

Roughly sketch $P(e^{j\omega})$ for $M = 4$, 8, and 16.

5.56. Let $x[m, n]$ be a signal that is a function of the two independent, discrete variables m and n. In analogy with one dimension and with the continuous-time case treated in Problem 4.53, we can define the two-dimensional Fourier transform of $x[m, n]$ as

$$X(e^{j\omega_1}, e^{j\omega_2}) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} x[m, n]e^{-j(\omega_1 m + \omega_2 n)}. \quad \text{(P5.56–1)}$$

(a) Show that eq. (P5.56–1) can be calculated as two successive one-dimensional Fourier transforms, first in m, with n regarded as fixed, and then in n. Use this result to determine an expression for $x[m, n]$ in terms of $X(e^{j\omega_1}, e^{j\omega_2})$.

(b) Suppose that

$$x[m, n] = a[m]b[n],$$

where $a[m]$ and $b[n]$ are each functions of only one independent variable. Let $A(e^{j\omega})$ and $B(e^{j\omega})$ denote the Fourier transforms of $a[m]$ and $b[n]$, respectively. Express $X(e^{j\omega_1}, e^{j\omega_2})$ in terms of $A(e^{j\omega})$ and $B(e^{j\omega})$.

(c) Determine the two-dimensional Fourier transforms of the following signals:

(i) $x[m, n] = \delta[m - 1]\delta[n + 4]$

(ii) $x[m, n] = (\frac{1}{2})^{n-m}u[n - 2]u[-m]$

(iii) $x[m, n] = (\frac{1}{2})^n \cos(2\pi m/3)u[n]$

(iv) $x[m, n] = \begin{cases} 1, & -2 < m < 2 \text{ and } -4 < n < 4 \\ 0, & \text{otherwise} \end{cases}$

(v) $x[m, n] = \begin{cases} 1, & -2 + n < m < 2 + n \text{ and } -4 < n < 4 \\ 0, & \text{otherwise} \end{cases}$

(vi) $x[m, n] = \sin\left(\frac{\pi n}{3} + \frac{2\pi m}{5}\right)$

(d) Determine the signal $x[m, n]$ whose Fourier transform is

$$X(e^{j\omega_1}, e^{j\omega_2}) = \begin{cases} 1, & 0 < |\omega_1| \le \pi/4 \text{ and } 0 < |\omega_2| \le \pi/2 \\ 0, & \pi/4 < |\omega_1| < \pi \text{ or } \pi/2 < |\omega_2| < \pi \end{cases}.$$

(e) Let $x[m, n]$ and $h[m, n]$ be two signals whose two-dimensional Fourier transforms are denoted by $X(e^{j\omega_1}, e^{j\omega_2})$ and $H(e^{j\omega_1}, e^{j\omega_2})$, respectively. Determine the transforms of the following signals in terms of $X(e^{j\omega_1}, e^{j\omega_2})$ and $H(e^{j\omega_1}, e^{j\omega_2})$:

(i) $x[m, n]e^{jW_1 m}e^{jW_2 n}$

(ii) $y[m, n] = \begin{cases} x[k, r], & \text{if } m = 2k \text{ and } n = 2r \\ 0, & \text{if } m \text{ is not a multiple of 2 or } n \text{ is not a multiple of 3} \end{cases}$

(iii) $y[m, n] = x[m, n]h[m, n]$