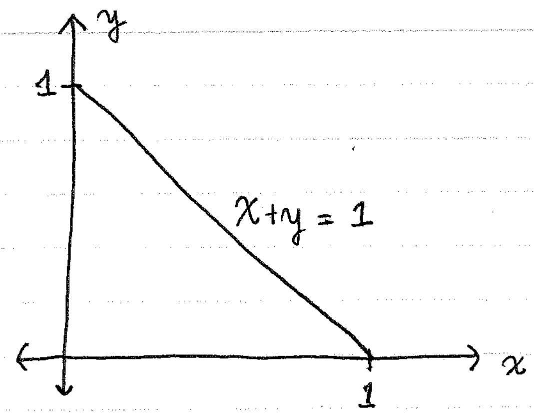
Ex

$$f_{X,Y}(x,y) = \begin{cases} 6x, & 0 \le x+y \le 1,\ x \ge 0,\ y \ge 0 \\ 0, & \text{else} \end{cases}$$

a) Find $E[X]$, $E[Y]$, $Var[X]$, $Var[Y]$, $Cov[X,Y]$



$$E[X] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x \, f_{X,Y}(x,y)\, dx\, dy$$

$$= \int_0^1 \int_0^{1-y} x \, 6x \, dx\, dy$$

$$= \int_0^1 2x^3 \Big|_0^{1-y} dy$$

$$= 2\int_0^1 (1-y)^3\, dy$$

$$= 2\int_0^1 u^3\, du = 1/2$$

$$E[Y] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y \, f_{X,Y}(x,y)\, dx\, dy$$

$$= \int_0^1 \int_0^{1-y} y \, 6x \, dx\, dy = 1/4$$

$$E[X^2] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^2 \, f_{X,Y}(x,y)\, dx\, dy$$

$$= \int_0^1 \int_0^{1-y} x^2 \, 6x \, dx\, dy = 3/10$$

$$E[Y^2] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y^2 f_{X,Y}(x,y)\,dx\,dy$$

$$= \int_0^1 \int_0^{1-y} y^2\, 6x \,dx\,dy = 1/10$$

$$\mathrm{Var}[X] = E[X^2] - (E[X])^2$$

$$= 3/10 - \left(\tfrac{1}{2}\right)^2 = 1/20$$

$$\mathrm{Var}[Y] = E[Y^2] - (E[Y])^2$$

$$= 1/10 - (1/4)^2 = 3/80$$

$$E[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\,y\, f_{X,Y}(x,y)\,dx\,dy$$

$$= \int_0^1 \int_0^{1-y} x\,y\, 6x \,dx\,dy = 1/10$$

$$\mathrm{Cov}[X,Y] = E[XY] - E[X]\,E[Y]$$

$$= 1/10 - \tfrac{1}{2}\left(\tfrac{1}{4}\right) = -1/40$$

b) Are X, Y independent, uncorrelated, or orthogonal?

- X, Y are not independent because we cannot write $f_{X,Y}(x,y) = a(x)\,b(y)$ due to the constraint $0 \le x + y \le 1$.

$\mathrm{Cov}[X,Y] \neq 0 \Rightarrow X, Y$ are not uncorrelated

$E[XY] \neq 0 \Rightarrow X, Y$ are not orthogonal

Properties of Mean, Variance, Covariance :

Let X, Y, U, V be r.v.s and a, b, c, d be constants.

1) $E[aX+b] = a\,E[X] + b$

2) $E[X+Y] = E[X] + E[Y]$

3) $Var[X] = E[X^2] - (E[X])^2$

4) $Var[aX+b] = a^2\,Var[X]$

5) $Var[X+Y] = Var[X] + Var[Y] + 2\,Cov[X,Y]$

Pf:
$$\begin{aligned}
Var[X+Y] &= E[(X+Y-E[X+Y])^2] \\
&= E[(X-E[X]+Y-E[Y])^2] \\
&= E[(X-E[X])^2] + E[(Y-E[Y])^2] \\
&\quad + 2\,E[(X-E[X])(Y-E[Y])] \\
&= Var[X] + Var[Y] + 2\,Cov[X,Y]
\end{aligned}$$

6) $Cov[X,Y] = E[XY] - E[X]E[Y]$

7) $Cov[aX+\overset{c}{\cancel{b}},\ bY+d] = ab\,Cov[X,Y]$

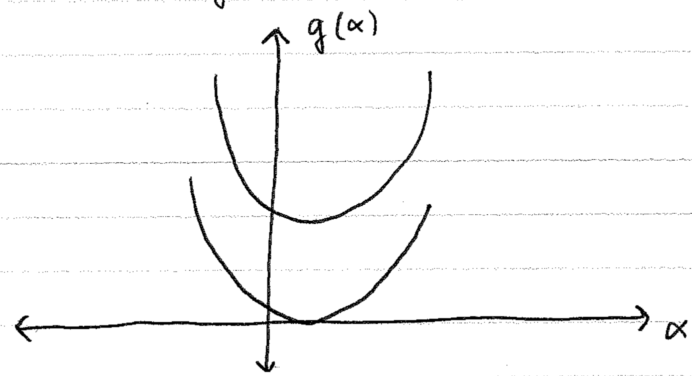
8) $Cov[X+Y,\ U+V] = Cov[X,U] + Cov[X,V] + Cov[Y,U] + Cov[Y,V]$

9) $|E[XY]| \leq \sqrt{E[X^2]E[Y^2]}$

Pf: Let $g(\alpha) = E[(\alpha X + Y)^2]$

$$= \alpha^2 E[X^2] + 2E[XY]\alpha + E[Y^2] \geq 0$$

Hence $g(\alpha)$ has at most one real root



$\Rightarrow$ $g(\alpha)$ has a nonpositive discriminant

$$b^2 - 4ac = (2E[XY])^2 - 4E[X^2]E[Y^2] \leq 0$$

$$\Rightarrow E[XY]^2 \leq E[X^2]E[Y^2]$$

$$\Rightarrow |E[XY]| \leq \sqrt{E[X^2]E[Y^2]}$$

10) $|Cov[X,Y]| \leq \sqrt{Var[X]\,Var[Y]}$

Pf Use 9) with $X' = X - \mu_X$, $Y' = Y - \mu_Y$

The magnitude of the covariance of X, Y is affected by the variances of X and Y. Because of this, normalization is used to talk about correlation on an absolute scale.

The <u>correlation coefficient</u> of X and Y is given by:

$$\rho_{XY} = \frac{Cov[X,Y]}{\sqrt{Var[X]\,Var[Y]}}$$

<u>Note</u>: From 10), $-1 \le \rho_{XY} \le 1$

<u>Ex</u>: Let X and Y be independent Gaussian r.v.s with jpdf:

$$f_{X,Y}(x,y) = \frac{1}{2\pi \cdot 144} \exp\left(\frac{-(x+1)^2}{18} + \frac{-(y+3)^2}{32} \right)$$

a) Find the mean and variance of X and Y.

X and Y are independent

$\Rightarrow f_{X,Y}(x,y) = f_X(x)\, f_Y(y)$

where $f_X(x)$, $f_Y(y)$ are Gaussian pdfs.

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma_X^2}} \exp\left(-\frac{(x-\mu_X)^2}{2\sigma_X^2}\right), \quad f_Y(y) = \frac{1}{\sqrt{2\pi\sigma_Y^2}} \exp\left(-\frac{(y-\mu_Y)^2}{2\sigma_Y^2}\right)$$

$$f_{X,Y}(x,y) = \frac{1}{2\pi \cdot 144} \exp\left(-\frac{(x-1)^2}{18}\right) \exp\left(-\frac{(y+3)^2}{32}\right)$$

$\Rightarrow \mu_X = 1 \quad , \quad \mu_Y = -3$

$\Rightarrow 2\sigma_X^2 = 18 \Rightarrow \sigma_X^2 = 9 \quad , \quad 2\sigma_Y^2 = 32 \Rightarrow \sigma_Y^2 = 16$

Can check that $\sqrt{2\pi\sigma_X^2} \cdot \sqrt{2\pi\sigma_Y^2} = 2\pi \cdot 144$

b) Let $U = 2X + Y$, $V = 3X - 2Y$
Find the mean and variance of U and V.

$$E[U] = E[2X+Y] = 2\cdot\mu_X + \mu_Y = -1 \quad , \quad E[V] = E[3X-2Y] = 3\mu_X - 2\mu_Y = 9$$

$$\begin{aligned} Var[U] &= Var[2X+Y] \\ &= Var[2X] + Var[Y] + 2\,Cov[2X, Y] \\ &= 4\sigma_X^2 + \sigma_Y^2 + 2\cdot 2\,\cancel{Cov[X,Y]}^{\,0} \\ &= 4\cdot 9 + 16 = 52 \end{aligned}$$

$$\begin{aligned} Var[V] &= Var[3X - 2Y] \\ &= Var[3X] + Var[-2Y] + 2\,Cov[3X, -2Y] \\ &= 9\,Var[X] + 4\,Var[Y] + 2(-6)\,\cancel{Cov[X,Y]}^{\,0} \\ &= 9\cdot 9 + 4\cdot 16 = 145 \end{aligned}$$