3 April 2011

Planar Graphs

Thm: $v + f = e + 2$

$\forall$ planar representations of G (connected)

f = # of faces/regions

BIG QUESTION:

What graphs are planar?

Def: The degree of a "face"

= number of edges encountered as one travel along the boundary of the faces.

e.g.

imagine walking inside the graph. Every edge counts, including the edged you've already walked on.

i.e. 3 edges $\succ$ counts twice

deg = 11

Handshake Thm for Faces

If G is planar (drawn in the plane w/o crossing) then

$\sum \deg(\text{Face}) = 2 \times e$

returning to previous example, we see that the boundary edges that may seem to count once in degree counting is counted once we realize that region outside the pentagon is also a face.

= How can we know a graph is planar w/o drawing it?

Lemma If G is simple, connected, and planar, then:

$e \leq 3v - 6$

Proof:

G simple $\Rightarrow$ no loops, no multiple edges

$\Rightarrow$ faces have degree of at least 3

 deg 1

 deg 2

We conclude: $2e = \sum \deg(\text{faces}) \geq \sum_{\text{all faces}} (3) = 3f$ — NOT HELPFUL

WE DON'T WANT TO DRAW THE GRAPH... and count f.
If we can draw it, it would already be planar...

Recall Euler and his wonderful conclusion

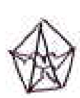
$$v + f = e + 2$$

that is, $f = e + 2 - v$

$$\Rightarrow \quad 2e \geq 3f = 3(e + 2 - v)$$

$$3v \geq e + 6 \qquad \square$$

Ex. $K = 5$



simple ✓
connected ✓
planar ?

$e = 10$ (choose 2 pairs of 10 vertices) $= \binom{5}{2}$

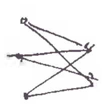
$f = 5$

By Lemma:

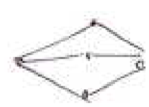
$$3(5) = 15 \geq 10 + 6 = 16 \quad \times$$

$k = 5$ is not planar

$K_{3,2}$



imagine flipping over to the other side $=$



obviously planar...

$K_{3,3}$

hmm planar?

$e = 9$ $\qquad 3(6) \geq 9 + 6$

$v = 6$ $\qquad 18 \geq 15$

Can we conclude something?

NO. It satisfies a planar condition.. but may not be planar

How can we come up w/ a stronger recipe?

Inspecting proof of Lemma:

We noted that all faces of a simple, connected, planar graph had degree ≥ 3

If one knew that G does not have cycles of length 3, then each face would have to have a degree of at least 4.

Improved Lemma:

G is connected, simple, planar, & no cycles of length 3.

then $\Rightarrow$ $v \geq \frac{e+4}{2}$

Pf: By handshake, we know

$$2e = \sum_{\text{all face}} \deg(f) \geq \sum_{\text{all face}} 4 = 4f = 4(e+2-v)$$

$$2e \geq 4e + 8 - 4v$$

$$4v \geq 2e + 8$$

$$v \geq \frac{e+4}{2} \qquad \square$$

Back to $K_{3,3}$

$e = 9 \quad v = 6$:

since $K_{3,3}$ is bipartite

$\rightarrow$ a path/cycle must have length > 3 else the path ends up in the wrong side.

thus,

$6 \geq \frac{9+4}{2}$ is false and $K_{3,3}$ ~~is not planar~~

So far, we showed that K_5 & $K_{3,3}$ is not planar

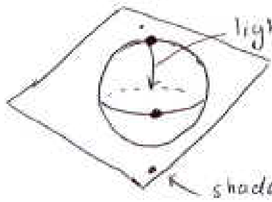
Thm (Kuratowski)

G is simple, connected, ...

G planar $\Leftrightarrow$ G does not contain K_5 or K_{33} as its subgraphs ...

Remark 1

stereographic projections

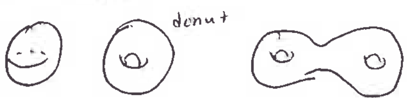


we see a 1-1 correspondance between points of $\mathbb{R}^2$ and points of $S^2 \setminus \{\text{noth pole}\}$.

$\Rightarrow$ G that can be drawn in $\mathbb{R}^2$ $\Rightarrow$ can be drawn in S

S^2 is smooth & compact (any $N=k$ has accumulation points)

other surfaces include:



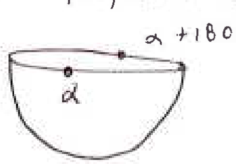
they are orientable surfaces

an non-orientable surface
"Klein bottle"

Q: Is there a version of kuratowski's Theorem that describes a list of forbidden graphs that indicate than a G cannot be drawn out on other surfaces ???

Another non-orientable surface

Real projective space



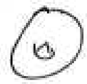
$\mathbb{R}\mathbb{P}^2$
identify opposite points out the boundary of a half sphere

There is a version of kuratowski Theorem on $\mathbb{R}\mathbb{P}^2$

$\rightarrow$ list of forbidden graph numbering ≈ 35

If a graph contains any of the forbidden graphs, it cannot be drawn on $\mathbb{R}\mathbb{P}^2$

∘) The list of forbidden graph is not known for

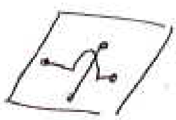


nor do we know if such list can be compiled.

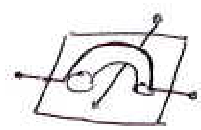
Remark 2

If you want to draw G on some smooth, compact space, it can be done more easily w/ increasing number of holes.

suppose



→ drilling and drawing a hole creates an easy passage.



more holes = more bridges.

Ultimate Lemma

G is simple, connected, planar, no circuit w length < k

$$\Rightarrow \quad v \geq \frac{k-2}{k} e + 2$$

Prf

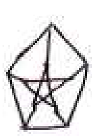
$$2e = \sum_{\text{faces}} \deg(\text{face}) \geq \sum_{\text{faces}} k = kf$$

$$\geq kf = k(e+2-v)$$

$$kv \geq (k-2)e + 2k$$

$$v \geq \frac{k-2}{k} e + 2 \quad \square$$

ex Peterson Graph.



Only cycles of length ≥ 5

$\Rightarrow K_5$

$$10 = v \geq \frac{5-2}{5} e + 2 = \frac{3}{5} \cdot 15 + 2 = 11 \quad \times$$