1. Assume that $1 < p < \infty$, $\dfrac{1}{p} + \dfrac{1}{q} = 1$, $f \in L^p(\mathbb{R})$ and $g \in L^q(\mathbb{R})$.
   (a) Let $h(x) = \int f(x-y)g(y)dy$. Show that h is bounded.
   (b) Show that h is continuous.

2. Suppose $\int_a^b f d\phi$ exists and is a Riemann-Stieltjes integral. Show
$$\int_a^b f d\phi = f(b)\phi(b) - f(a)\phi(a) - \int_a^b \phi df$$
.

3. For what α does $\sum_{n=1}^{\infty} \sin^{\alpha}(n! e\pi)$ converge absolutely?

4. Evaluate the following limits.
   (a)
$$\lim_{t \to 0} \int_0^1 \frac{e^{-t \ln x} - 1}{t} dx$$
   (b)
$$\lim_{n \to \infty} \int_1^{n^2} \frac{n \cos(\frac{x}{n^2})}{1 + n \ln x} dx$$

5. Let $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$. Assume that $f \in L^1(\mathbb{R}_{\geq 0})$ satisfies
$$f(x) \leq c \int_0^x f(t) dm(t)$$
with c independent of x. Show that $f(x) = 0$ for every $x \in \mathbb{R}_{\geq 0}$.

6. Find all functions $f \in BV[0,1]$ with $\int_0^1 f = \frac{1}{3}$, and
$$f(0) - (V_0^x)^{\frac{1}{2}} = f(x).$$

7. Let μ be a finite measure on A, let f be measurable and bounded on A, and let ϕ be convex in an interval containing the range of f. Prove that
$$\phi\left(\frac{\int_A f \, d\mu}{\int_A d\mu}\right) \leq \frac{\int_A \phi(f) d\mu}{\int_A d\mu}.$$

8. Suppose $f_n, n \in \mathbb{N}$, is a sequence of nonnegative measurable functions on a measure space $(\Omega, \mathcal{A}, \mu)$, $f_n \to f$ a.e., and there are subsequences f_{n_j}, f_{m_j} such that
$$\int_{\Omega} f_{n_j} \to 1, \quad \int_{\Omega} f_{m_j} \to 3 \text{ as } j \to \infty.$$
State and justify the **best** upper and lower bounds for $\int_{\Omega} f$ based on the given information.

9. Let $f \in L^1([0,1])$ and let $F(x) = \int_0^x f(t)dt$. If E is a measurable subset of $[0,1]$, show that

   (a) $F(E) = \{y : \exists x \in E \text{ with } y = F(x)\}$ is measurable.

   (b) $m(F(E)) \leq \int_E |f(t)|dt$.