LAB #7

Resonance

Goal: Observe the phenomenon of resonance; find numerical approximations of solutions to non-autonomous systems of differential equations.

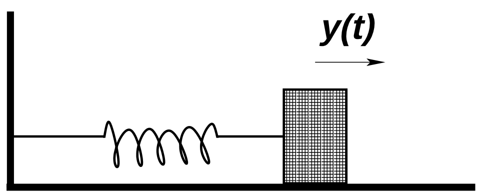
Required tools: MATLAB routines ***pplane*** , **ode45** ; m-files; systems of differential equations.

Discussion

Assume that we have a box of mass 1g on a table and attached to a spring. Initially, the spring is unstretched. We pull the box 3 cm to the right and give it an initial speed of 1 cm/sec to the left. From Newton's 2^{nd} Law $F = ma$, we have seen that the motion of the spring is governed by the equation

$$my'' + \mu y' + ky = 0 \qquad (*)$$

where m is the mass of the box, μ is the coefficient of friction and k is the spring constant.



Assignment

(1) Assume that $m = 1$, $\mu = 0$, and $k = 0.25$ dynes/cm. Thus, it takes 0.25 dynes of force to stretch the spring 1 cm. Convert equation $(*)$ to a system, enter the system into ***pplane*** . Use the "Keyboard Input" option to plot the phase plane portrait corresponding to a 3 cm stretching with initial velocity -1 cm/sec. Under the "Graph" pull down menu select "Plot y vs t". Use this graph to estimate the period of the motion. Would you consider this slow or fast oscillation? (The time is in seconds.) Use your graph to estimate the amplitude of the motion.

(2) Find the general solution to $(*)$, with the values of μ and k from (1). Find a formula for the solution that satisfies $y(0) = 3$ and $y'(0) = -1$. Use your answer to find the exact value for the period and amplitude of the motion you estimated in (1).

(3) The period of the oscillations is determined by the "stiffness" of the spring. (A stiff spring is one which takes a large force to stretch it.) How do you guess stiffness should relate the period of the motion. i.e., should stiff springs oscillate faster or slower ? Test your guess by graphing the solution curve for a stiff spring and a non-stiff spring. Note: This will require that you change k in equation$(*)$. How should you change it to model a stiffer spring?

(4) Imagine that our box is resting near your stereo that is generating a tone which is causing the box to vibrate. Assume that the box is initially at rest and any motion is due entirely to vibrations of your stereo (hence $y(0) = y'(0) = 0$). We can model this as applying an external force of $F(t) = \left(\frac{\textbf{seed}}{10}\right)\sin(\omega t)$, where ω is determined by the pitch of the tone. Now equation $(*)$ becomes

$$my'' + \mu y' + ky = \left(\frac{\textbf{seed}}{10}\right)\sin(\omega t) \qquad (**)$$

Assume that $m = 1, k = 0.25$, and $\mu = 0$. The equivalent system is then

$$\begin{cases} y' = v \\ v' = -0.25y + \left(\frac{\textbf{seed}}{10}\right)\sin(\omega t) \end{cases} \qquad (***)$$

where $y(0) = 0$ and $v(0) = 0$. The above system $(***)$ is very different from the system corresponding to equation $(*)$ due to the explicit presence of t in the second equation. Thus $(***)$ is an example of a *non-autonomous* system while the system corresponding to equation $(*)$ is autonomous. In this part of the exercise, we want to plot the solution as a function of t for several different values of ω. Unfortunately, ***pplane*** only applies to an <u>autonomous</u> system. Thus, we will use the MATLAB routine **ode45**. For this, you must first create a function file (an *m-file*) to represent the right side of this equation. You would first create an m-file called, say, "yvp.m" containing the lines

```
function x=yvp(t,u);

global w;

x=[u(2); -0.25*u(1)+(seed/10)*sin(w*t)];
```

Save this file as **yvp.m**. Note that $u(1)$ represents y; $u(2)$ represents v; and x represents $[y', v']$. The "global" command will allow you to change the value of w without having to continually edit the m-file itself.

To solve the differential equation for various ω and plot the solution y vs t, just enter:

```
>>global w;
>>w=0.1; [t,u]=ode45('yvp',[0, 40],[0, 0]); plot(t,u(:,1));
```

This tells MATLAB to solve the equation for $0 \le t \le 40$ when $\omega = 0.1$. The expression `[0,0]` represents the initial position and the initial velocity, respectively. The output matrix `[t, u]` is a matrix with 3 columns. The 1^{st} column is a vector of time values; the 2^{nd} column is a vector of the corresponding displacements y; the 3^{rd} column is a vector of the corresponding velocities v. The last expression above will plot the displacements y vs t. (If we wanted to plot the velocities, use `u(:,2)`.)

Use the above to plot the graphs of y for $\omega = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8$, and 0.9 (use the $\uparrow$ key to easily change the values of ω). *Don't print these graphs,* however in each case use your graph to determine (approximately) the maximum displacement of the box from its rest position. To find the maximum, you may need to graph y over a larger time interval. Construct (either by hand or on the computer) a graph of the maximum displacement as a function of ω.

Note: For better accuracy, you might want to put more tick marks onto the y-axis than MATLAB's default option provides. The following command for example puts marks from -10 to 20 at 0.5 intervals. It should be executed after the graph is drawn. You could put it on the same line as the plot command to ease re-executing it.) You might also want to execute "grid on" which will draw a grid on the graph.

```
>> set(gca,'Ytick',-10:0.5:20);
```

Remark. The *frequency* of a wave is the number of oscillations per unit period of time. The period is the time it takes to complete each oscillation. It follows that the frequency is the reciprocal of the period. Hence, the frequency of $\sin(\omega t)$ is $\frac{\omega}{2\pi}$ cycles per second. Thus, ω is proportional to the frequency. The graph of maximum displacement against frequency is referred to as the *frequency response* of the system. It indicates how sensitive the system is to changes in the driving frequency. Your graph is essentially a graph of the frequency response of the box.

(5) In Part (4), you should find that the behavior at $\omega = 0.5$ is quite extraordinary. What you are observing is a phenomena call **RESONANCE**. If we vibrate the box with just the right frequency, the oscillations become ever bigger and bigger. Prove that this is indeed correct by solving equation $(**)$ with the initial conditions $y(0) = 0, y'(0) = 0$ and $\omega = 0.5$. Notice that the peaks in the graph for this case seemed to lie on a straight line. Use your solution to prove that this is true. What is the slope of this line? The frequency at which resonance occurs is called the *resonant frequency* of the system. Hence, the resonant frequency of our box is $\frac{0.5}{2\pi}$ cycles per second. In general, the resonant frequency for a system described by $y'' + ky = 0$ is $\frac{\sqrt{k}}{2\pi}$ which is also the frequency at which the system oscillates without any external force.

(6) The behavior you observed in Part (4) would not happen in real life. One limiting factor is friction. Assume that in equation $(*)$ $\mu = 0.3$. Construct a graph of the frequency response of the new system as you did in Part (4). Try to estimate as accurately as you can the value of ω for which the maximum displacement is greatest (note: $\omega \neq 0.5$).

Note: The maximum does not change rapidly as ω moves away from 0.5. To judge the place where the maximum occurs, you may find it convenient to graph the curves for several different values of ω, in different colors, on the same graph so that you can tell which curve is highest. You might also want to have grid on.

Turn in only the graph of the frequency response with the maximum displacement indicated.

The value of $\frac{\omega}{2\pi}$ at which the maximum occurs would be the resonant frequency for this system.

(7) In your write up, discuss what changing the spring constant does to the rate at which the spring oscillates. What is resonance? Discuss the effect of resistance on resonance. Does adding resistance change the resonant frequency? If so, does the frequency increase or decrease? In this lab, we only investigated two values of μ, namely $\mu = 0.0$ and $\mu = 0.3$. As a guess, what do you think a plausible frequency response curve would look like for a considerably smaller, but non-zero, value of μ ? Draw, on one sheet of paper, the graphs from Part (4), Part (6), and your guess. Indicate very carefully, where you expect the maximum of your guess to lie. Explain why you put it there.