MA598A - Summer 2013Problem Set #2 06.12.13 (1)
Group Theory, II

1. Suppose that G is a group and that the set
$$\{x \in G \mid |x| = 2\}$$
has exactly one element. Show that G is abelian.

2. You are given that G is group of order 24 which is not isomorphic to S_4. Show that one of its Sylow subgroups is normal.

3. Determine the number of pairwise non-isomorphic groups of order pq, where p and q are prime.

4. Let $\varphi : G \to H$ be a homomorphism of groups. Let $G^{\#}$ and $H^{\#}$ denote the set of conjugacy classes in G and H, respectively.
   (a) Show that φ induces a map $\varphi^{\#} : G^{\#} \to H^{\#}$.
   (b) Show that if $\varphi^{\#}$ is injective, so is φ.
   (c) Show that if $\varphi^{\#}$ is surjective, and H is finite, then φ is surjective. (*Hint: one of the problems from PS1 might be useful here...*)

5. Let G be a group of order 56 with a normal 2-Sylow subgroup Q, and let P be a 7-Sylow subgroup of G. Show that $G \cong P \times Q$ or $Q \cong \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$.

6. Let G be a group and H a subgroup of G with finite index. Show that there exists a *normal* subgroup N of G of finite index with $N \subset H$.

7. Let G be a finite group and P a p-Sylow subgroup of G for some prime p. (You may assume that p divides $|G|$. I haven't had enough coffee to think about the implications of the vacuous case...)
   (a) Assume p=2 and P is cyclic. Show that the normalizer and centralizer of P coincide.
   (b) Show that this may not hold if $p = 2$ but P is not cyclic.
   (c) Show that the first statement does not hold regardless of cyclicity if $p \neq 2$.

8. Let G be a finite group and $\varphi : G \to G$ a homomorphism. Show that $\varphi(P)$ is a subgroup of P whenever P is a normal Sylow subgroup.

9. (a) Find all simple groups of order 101.
   (b) Find all simple groups of order 102.
   (c) Find all groups of order 175.

10. Let p and q be primes such that p divides $q-1$.

   (a) Show that there exists a group G with generators x and y and relations

   $$x^{p^2} = 1, \quad y^q = 1, \quad xyx^{-1} = y^a,$$

   where a is an integer not congruent to 1 mod q, but $a^p \cong 1(\operatorname{mod} q)$.

   (b) Prove that the Sylow q-subgroup $S_q \subset G$ is normal.

   (c) Prove that G/S_q is cyclic; and deduce that G has a unique subgroup H of order pq.

   (d) Prove that H is cyclic.

   (e) Prove that any subgroup of G with order p is contained in H, hence is generated by x^p and is contained in the center of G.

   (f) Prove that the center of G is the unique subgroup of G having order p.

   (g) Prove that every proper subgroup of G is cyclic.

   (h) For each positive divisor d of p^2q, determine the number of elements of G having order d.