(15 pts) **1.** Using the definition of the Fourier transform, compute the Fourier transform of the DT signal:

$$x[n] = e^{j\frac{\pi}{17}n}\left(\frac{1}{3}\right)^n u[n-1].$$

$$X(\omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n} = \sum_{n=-\infty}^{\infty} e^{j\frac{\pi}{17}n}\left(\tfrac{1}{3}\right)^n u[n-1]\, e^{-j\omega n} = \sum_{n=1}^{\infty} \left(\tfrac{1}{3}\right)^n e^{n(j\frac{\pi}{17}-j\omega)}$$

$$= \sum_{n=1}^{\infty} \left(\tfrac{1}{3} e^{j\pi/17-j\omega}\right)^n$$

Let $m = n-1 \Rightarrow n = m+1$

$$\Rightarrow X(\omega) = \sum_{m=0}^{\infty} \left(\tfrac{1}{3} e^{j\pi/17-j\omega}\right)^{m+1} = \tfrac{1}{3} e^{j\pi/17-j\omega} \sum_{m=0}^{\infty} \left(\tfrac{1}{3} e^{j\pi/17-j\omega}\right)^m$$

$$= \tfrac{1}{3} e^{j\pi/17-j\omega}\left(\frac{1}{1-\frac{1}{3}e^{j\pi/17-j\omega}}\right) = \frac{1}{3e^{j\omega-j\pi/17}-1}$$

could stop here

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(15 pts) **2.** Using the definition of the inverse Fourier transform, compute the CT inverse Fourier transform of

$$\mathcal{X}(\omega) = \sum_{k=-\infty}^{\infty} \frac{1}{2^k} u[k] \delta(\omega - k\pi).$$

$$x[n] = \frac{1}{2\pi} \int_{2\pi} \mathcal{X}(w) e^{jwn} dw = \frac{1}{2\pi} \int_{2\pi} e^{jwn} \sum_{K=-\infty}^{\infty} \frac{1}{2^K} u[K] \delta(w - K\pi) dw$$

$$= \frac{1}{2\pi} \int_{2\pi} e^{jwn} \sum_{K=0}^{\infty} \left(\tfrac{1}{2}\right)^K \delta(w - K\pi) dw = \frac{1}{2\pi} \sum_{K=0}^{\infty} \left(\tfrac{1}{2}\right)^K \int_0^{2\pi} e^{jwn} \delta(w - K\pi) dw$$

$$= \frac{1}{2\pi} \sum_{K=0}^{\infty} \left(\tfrac{1}{2}\right)^K e^{jK\pi n} = \frac{1}{2\pi} \sum_{K=0}^{\infty} \left(\tfrac{1}{2} e^{j\pi n}\right)^K = \frac{1}{2\pi} \cdot \frac{1 - \cancel{\left(\tfrac{1}{2} e^{j\pi n}\right)^{2\pi}}}{1 - \tfrac{1}{2} e^{j\pi n}}$$

$$= \frac{1}{2\pi} \cdot \frac{1 - \tfrac{1}{8} e^{j3\pi n}}{1 - \tfrac{1}{2} e^{j\pi n}}$$

Stop here

$$\rightarrow x[n] = \begin{cases} \frac{1}{2\pi} \cdot \frac{7/8}{1/2}, & \text{for even } n \\ \frac{1}{2\pi} \cdot \frac{9/8}{3/2}, & \text{for odd } n \end{cases}$$

$$= \begin{cases} \frac{7}{8\pi}, & \text{for even } n \\ \frac{3}{8\pi}, & \text{for odd } n \end{cases}$$

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cosine of a real signal is real $\Rightarrow$ $j\{\text{Real}\}$ is pure imaginary.
cosine of a real signal is even $\Rightarrow$ $x[n]$ is even.

$\rightarrow$ $x[n]$ is pure imaginary and even.

(15 pts) 3. Given is a DT signal $x[n] = j\cos(g[n])$ where $g[n]$ is a real signal and an odd function of n.

a) Bob claims that the Fourier transform of $x[n]$ is $\mathcal{X}(\omega) = \frac{j}{\sin\omega}$. Explain why Bob's answer is wrong.

$\mathcal{X}^*(-\omega) = \frac{-j}{\sin -\omega} = \frac{j}{\sin\omega} = \mathcal{X}(\omega) \rightarrow x[n]$ is real (by 43), but the given $x[n]$ is pure imaginary $\rightarrow$ this is not a possible $\mathcal{X}(\omega)$.

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b) Alice says that the Fourier transform of $x[n]$ is $\mathcal{X}(\omega) = \frac{3}{\cos\omega}$. Could Alice be right? Explain.

$\mathcal{X}(-\omega) = \frac{3}{\cos(-\omega)} = \frac{3}{\cos\omega} = \mathcal{X}(\omega) \rightarrow \mathcal{X}(\omega)$ is even

$\mathcal{X}(\omega)$ is also real $\rightarrow$ (by 44.) $x[n]$ is real and even, but the given $x[n]$ is pure Imaginary $\Rightarrow$ this is not a possible $\mathcal{X}(\omega)$.

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c) Devin says that the Fourier transform of $x[n]$ is $\mathcal{X}(\omega) = \frac{j}{(\omega^2+1)^2}$. Could Devin be right? Explain.

0

$\sum_{n=-\infty}^{\infty} |j\cos(g[n])|^2 = \sum_{n=-\infty}^{\infty} \cos^2(g[n])$, which is finite for say $g[n] = \begin{cases} a & n=1 \\ -a & n=-1 \\ 0 & \text{, else} \end{cases}$ (for some finite real a)

$\frac{1}{2\pi}\int_{2\pi} \left|\frac{j}{(\omega^2+1)^2}\right|^2 d\omega = \frac{1}{2\pi}\int_{2\pi} \frac{1}{(\omega^2+1)^4} d\omega$ is finite

$\rightarrow$ Parseval's Relation could be satisfied for some a.

$\mathcal{X}^*(-\omega) = \frac{-j}{((-\omega)^2+1)^2} = \frac{-j}{(\omega^2+1)^2} = -\mathcal{X}(\omega) \rightarrow x[n]$ is not pure real.

$\Rightarrow$ Devin could be right.

No because the F.T of a DT signal is always periodic.

5

4. A discrete-time LTI system is defined by the difference equation

$$y[n] - \frac{3}{4}y[n-1] + \frac{1}{8}y[n-2] = 2x[n]$$

(10 pts) a) Find the frequency response of this system. (Use the properties of the Fourier transform listed in the table to justify your answer. Do not just plug into a formula you have learned by heart.)

$$\mathcal{F}\left(y[n] - \tfrac{3}{4}y[n-1] + \tfrac{1}{8}y[n-2]\right) = \mathcal{F}(2x[n])$$

$$\begin{pmatrix}\text{By}\\ 33+34\end{pmatrix} \Rightarrow Y(\omega) - \tfrac{3}{4}e^{-j\omega}Y(\omega) + \tfrac{1}{8}e^{-j\omega 2}Y(\omega) = 2X(\omega)$$

$$\Rightarrow Y(\omega)\left(1 - \tfrac{3}{4}e^{-j\omega} + \tfrac{1}{8}e^{-j\omega 2}\right) = 2X(-\omega)$$

$$\Rightarrow \mathcal{H}(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{2}{1 - \frac{3}{4}e^{-j\omega} + \frac{1}{8}e^{-j\omega 2}} \times \frac{8}{8} = \frac{16}{(e^{-j\omega})^2 - 6e^{-j\omega} + 8} = \frac{16}{(e^{-j\omega}-2)(e^{-j\omega}-4)}$$

$$= \frac{A}{e^{-j\omega}-2} + \frac{B}{e^{-j\omega}-4}$$

stop here

$$\Rightarrow A = \frac{16}{e^{-j\omega}-4}\bigg|_{e^{-j\omega}=2} = \frac{16}{2-4} = \frac{16}{-2} = -8$$

$$\Rightarrow B = \frac{16}{e^{-j\omega}-2}\bigg|_{e^{-j\omega}=4} = \frac{16}{4-2} = \frac{16}{2} = 8$$

goes on scratch paper

$$\Rightarrow \mathcal{H}(\omega) = \frac{-8}{e^{-j\omega}-2} + \frac{8}{e^{-j\omega}-4} = \frac{8}{2-e^{-j\omega}} - \frac{8}{4-e^{-j\omega}} = \frac{4}{1-\frac{1}{2}e^{-j\omega}} - \frac{2}{1-\frac{1}{4}e^{-j\omega}}$$

10

(10 pts) b) What is the unit impulse response of this system. (Justify your answer)

$$h[n] = \mathcal{F}^{-1}(\mathcal{H}(\omega)) = \mathcal{F}^{-1}\left(\frac{4}{1-\frac{1}{2}e^{-j\omega}} - \frac{2}{1-\frac{1}{4}e^{-j\omega}}\right) \overset{(\text{By } 33)}{=} 4\,\mathcal{F}^{-1}\left(\frac{1}{1-\frac{1}{2}e^{-j\omega}}\right) - 2\,\mathcal{F}^{-1}\left(\frac{1}{1-\frac{1}{4}e^{-j\omega}}\right)$$

$$\overset{(\text{By } 31)}{=} 4\left(\tfrac{1}{2}\right)^n u[n] - 2\left(\tfrac{1}{4}\right)^n u[n]$$

10

(5 pts) c) What is the Fourier transform of the output when the input is $x[n] = \left(\frac{1}{4}\right)^n u[n]$? (Justify your answer.)

$$\mathcal{X}(\omega) = \mathcal{F}(x[n]) = \mathcal{F}\left(\left(\tfrac{1}{4}\right)^n u[n]\right) \overset{(\text{By } 31)}{=} \frac{1}{1-\frac{1}{4}e^{-j\omega}}$$

$$\mathcal{Y}(\omega) = \mathcal{H}(\omega)\,\mathcal{X}(\omega) = \left(\frac{4}{1-\frac{1}{2}e^{-j\omega}} - \frac{2}{1-\frac{1}{4}e^{-j\omega}}\right)\frac{1}{1-\frac{1}{4}e^{-j\omega}}$$

stop here

$$= \frac{4}{(1-\frac{1}{2}e^{-j\omega})(1-\frac{1}{4}e^{-j\omega})} - \frac{2}{(1-\frac{1}{4}e^{-j\omega})^2}$$

goes on scratch

$$= \frac{A}{1-\frac{1}{2}e^{-j\omega}} + \frac{B}{1-\frac{1}{4}e^{-j\omega}} - \frac{2}{(1-\frac{1}{4}e^{-j\omega})^2}$$

$$\Rightarrow A = \left.\frac{4}{1-\frac{1}{4}e^{-j\omega}}\right|_{e^{-j\omega}=2} = \frac{4}{1-\frac{1}{2}} = \frac{4}{\frac{1}{2}} = 8$$

$$\Rightarrow B = \left.\frac{4}{1-\frac{1}{2}e^{-j\omega}}\right|_{e^{-j\omega}=4} = \frac{4}{1-2} = \frac{4}{-1} = -4$$

$$\Rightarrow \mathcal{Y}(\omega) = \frac{8}{1-\frac{1}{2}e^{-j\omega}} - \frac{4}{1-\frac{1}{4}e^{-j\omega}} - \frac{2}{(1-\frac{1}{4}e^{-j\omega})^2}$$

5

(10 pts) d) What is the output when the input is $x[n] = \left(\frac{1}{4}\right)^n u[n]$? (Justify your answer)

$$y[n] = \mathcal{F}^{-1}(Y(w)) = \mathcal{F}^{-1}\left(\frac{8}{1-\frac{1}{2}e^{-jw}} - \frac{4}{1-\frac{1}{4}e^{-jw}} - \frac{2}{(1-\frac{1}{4}e^{-jw})^2}\right)$$

$$\overset{(By\ 33)}{=} 8\,\mathcal{F}^{-1}\left(\frac{1}{1-\frac{1}{2}e^{-jw}}\right) - 4\,\mathcal{F}^{-1}\left(\frac{1}{1-\frac{1}{4}e^{-jw}}\right) - 2\,\mathcal{F}^{-1}\left(\frac{1}{(1-\frac{1}{4}e^{-jw})^2}\right)$$

$$\overset{(By\ 31+32)}{=} 8\left(\frac{1}{2}\right)^n u[n] - 4\left(\frac{1}{4}\right)^n u[n] - 2(n+1)\left(\frac{1}{4}\right)^n u[n]$$

10

could skip these steps

(15 pts) 5. Use the properties of the Fourier transform to evaluate the following integral:

$$\int_{-\infty}^{\infty} \frac{\sin^2(4t)}{t^2}\,dt.$$

$$\int_{-\infty}^{\infty} \frac{\sin^2(4t)}{t^2}\,dt = \int_{-\infty}^{\infty} \left|\frac{\sin(4t)}{t}\right|^2 dt \overset{(\text{By } 20)}{=} \frac{1}{2\pi}\int_{-\infty}^{\infty} \left|\mathcal{F}\left(\frac{\sin(4t)}{t}\right)\right|^2 dw$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \left|\mathcal{F}\left(\pi\cdot\frac{\sin(4t)}{\pi t}\right)\right|^2 dw \overset{(\text{By } 9)}{=} \frac{\pi}{2}\int_{-\infty}^{\infty} \left|\mathcal{F}\left(\frac{\sin(4t)}{\pi t}\right)\right|^2 dw$$

$$\overset{(\text{By } 5)}{=} \frac{\pi}{2}\int_{-\infty}^{\infty} \left|u(w+4)-u(w-4)\right|^2 dw$$

$$= \frac{\pi}{2}\int_{-4}^{4} dw = \frac{\pi}{2}\left[w\right]_{-4}^{4} = \frac{\pi}{2}(4+4) = \frac{8}{2}\pi = 4\pi$$

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