3) Basic Properties and Results of DT conv.

- Properties of DT conv.

1. Commutativity : $x_1[n] * x_2[n] = x_2[n] * x_1[n]$
2. Associativity : $(x_1[n] * x_2[n]) * x_3[n]$
$= x_1[n] * (x_2[n] * x_3[n])$
3. Distributive : $x_1[n] * (x_2[n] + x_3[n])$
$= x_1[n] * x_2[n] + x_1[n] * x_3[n]$

Discrete-Time Conv. Examples.

Without explicitly doing conv., but rather using basic conv. results plus properties of conv.

– Basic Conv. Results.

(example 2.3) in text

$$\left.\begin{array}{l} x[n] = \alpha^n u[n] \\ h[n] = u[n] \end{array}\right\} \rightarrow y[n] = x[n] * h[n]$$

$$= \frac{1}{1-\alpha}\left(1-\alpha^{n+1}\right) u[n]$$

$$= \frac{1}{1-\alpha} u[n] - \frac{\alpha}{1-\alpha} \alpha^n u[n]$$

special case where $\beta = 1$

(Recall)

- $$\left.\begin{array}{l} x[n] = \alpha^n u[n] \\ h[n] = \beta^n u[n] \end{array}\right\} \rightarrow y[n] = x[n] * h[n]$$

$$= \frac{\beta}{\beta-\alpha} \cdot \beta^n u[n] - \frac{\alpha}{\beta-\alpha} \alpha^n u[n]$$

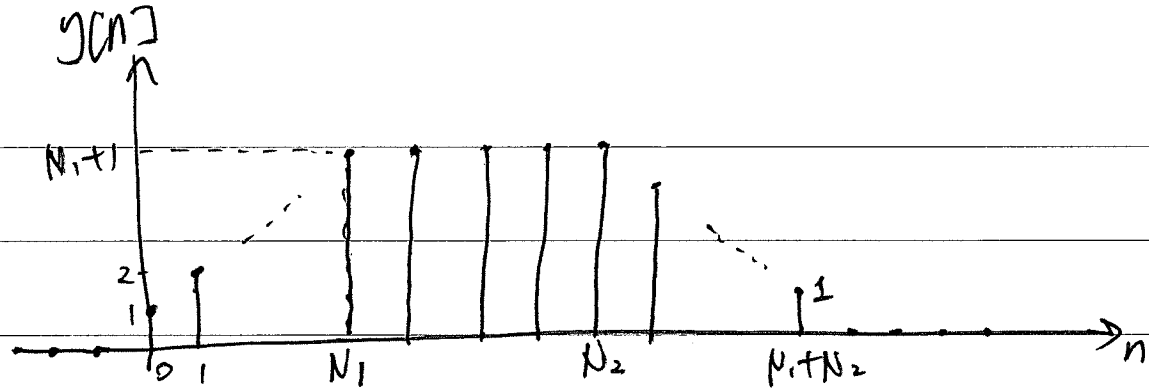
$$= \frac{1}{\beta-\alpha}\left(\beta^{n+1} - \alpha^{n+1}\right) u[n]$$

(other basic result) : conv. of two DT rectangles.

$$x[n] = u[n] - u[n-(N_1+1)] = \begin{cases} 1, & 0 \le n \le N_1 \\ 0, & o/w \end{cases} \quad : \text{length} = N_1+1$$

$$h[n] = u[n] - u[n-(N_2+1)] = \begin{cases} 1, & 0 \le n \le N_2 \\ 0, & o/w \end{cases} \quad : \text{length} = N_2+1$$

assume $N_2 \ge N_1$



Properties of Conv.

- (i) associativity (ii) commutativity (iii) distributive
- Recall and use concepts of time-invariance and linearity (homogeneity and superposition)

$\Rightarrow$ If $y[n] = x[n] * h[n]$, then

1. $z[n] = x[n-n_0] * h[n] = y[n-n_0]$
2. $z[n] = x[n] * h[n-n_0] = y[n-n_0]$
3. $z[n] = x[n-n_1] * h[n-n_2] = y[n-(n_1+n_2)]$

n_1, n_2 can be any integers.

e.g.

- Consider convolving two finite-length geometric sequences.

$$\left.\begin{aligned} x[n] &= \alpha^n \{u[n] - u[n-N_1]\} \\ h[n] &= \beta^n \{u[n] - u[n-N_2]\} \end{aligned}\right\} \rightarrow y[n] = x[n] * h[n] = ?$$

$$\rightarrow x[n] * h[n] = \underbrace{\{\alpha^n u[n] - \alpha^{N_1}\alpha^{n-N_1} u[n-N_1]\}}_{x[n]} * \underbrace{\{\beta^n u[n] - \beta^{N_2}\beta^{n-N_2} u[n-N_2]\}}_{h[n]}$$

$$= \left(\begin{aligned} &\alpha^n u[n] * \beta^n u[n] &&= \frac{1}{\beta-\alpha}(\beta^{n+1} - \alpha^{n+1}) u[n] \\ &- \alpha^n u[n] * \beta^{N_2}\beta^{n-N_2} u[n-N_2] &&= -\beta^{N_2} \cdot \frac{1}{\beta-\alpha}(\beta^{n+1-N_2} - \alpha^{n+1-N_2}) u[n-N_2] \\ &- \alpha^{N_1}\alpha^{n-N_1} u[n-N_1] * \beta^n u[n] &&= -\alpha^{N_1} \cdot \frac{1}{\beta-\alpha}(\beta^{n+1-N_1} - \alpha^{n+1-N_1}) u[n-N_1] \\ &+ \alpha^{N_1}\alpha^{n-N_1} u[n-N_1] * \beta^{N_2}\beta^{n-N_2} u[n-N_2] &&= \alpha^{N_1}\beta^{N_2} \frac{1}{\beta-\alpha}(\beta^{n+1-(N_1+N_2)} - \alpha^{n+1-(N_1+N_2)}) \\ & && \quad \times u[n-(N_1+N_2)] \end{aligned} \right)$$

$$\oplus \Rightarrow y(t)$$

~~We obtain~~ α, β are arbitrary complex-valued constants (in general)

N_1, N_2 are positive integers

We obtained a closed-form answer without having to explicitly compute a conv.

(Prob 2.26) $y[n] = \underline{x_1[n] * x_2[n]} * x_3[n]$

$x_1[n] = \left(\frac{1}{2}\right)^n u[n]$, $x_2[n] = u[n+3]$, $x_3[n] = \delta[n] - \delta[n-1]$

(a) $x_1[n] * x_2[n] = ?$

$\alpha = \frac{1}{2}$ $\beta = 1$

$n = n+3$

$$\frac{1}{\beta-\alpha}\left(\beta^{n+1} - \alpha^{n+1}\right) u[n]$$

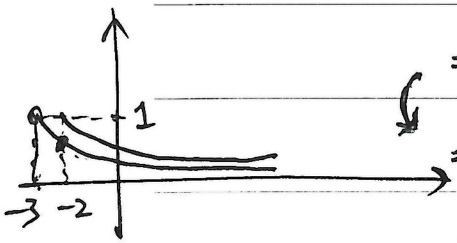
$$\rightarrow \frac{1}{\beta-\alpha}\left(\beta^{n+4} - \alpha^{n+4}\right) u[n+3]$$

$$= 2\left(1 - \left(\frac{1}{2}\right)^{n+4}\right) u[n+3]$$

$$= 2u[n+3] - \left(\frac{1}{2}\right)^{n+3} u[n+3] = z[n]$$

(b) $y[n] = z[n] * x_3[n] = z[n] * \{\delta[n] - \delta[n-1]\}$

$$= z[n] - z[n-1]$$



$$= \left\{2u[n+3] - \left(\frac{1}{2}\right)^{n+3} u[n+3]\right\} - \left\{2u[n+2] - \left(\frac{1}{2}\right)^{n+2} u[n+2]\right\}$$

$$= \cancel{2u[n+2]} + 2\delta[n+3] - \delta[n+3] - \left(\frac{1}{2}\right)^{n+3} u[n+2] - \cancel{2u[n+2]} + \left(\frac{1}{2}\right)^{n+2} u[n+2]$$

$$= \delta[n+3] - \left(\frac{1}{2}\right)^{n+3} u[n+2] + \left(\frac{1}{2}\right)^{n+2} u[n+2]$$

$$= \delta[n+3] + \underbrace{\left(-\frac{1}{8} + \frac{1}{4}\right)}_{=\frac{1}{8}} \left(\frac{1}{2}\right)^n u[n+2]$$

Prob 2.5 , 2.21 — Do it Yourself

4) Impulse Response ~~of~~ for simple first-order Difference Eq. "Find a pattern"

e.g. 1) $y[n] = a\,y[n-1] + x[n]$

Determine impulse response. (zero initial conditions)

$h[n] = 0$ for $n < 0$ (causal)

$h[n] = a\,h[n-1] + \delta[n]$: iterate thru recursively.

$n=0 \quad h[0] = a\,\underbrace{h[-1]}_{0} + \underbrace{\delta[0]}_{1} = \underline{1} = a^0$

$n=1 \quad h[1] = a\,\underbrace{h[0]}_{1} + \underbrace{\delta[1]}_{0} = a \cdot 1 = \underline{a^1}$

$n=2 \quad h[2] = a\,\underbrace{h[1]}_{a} + \underbrace{\delta[2]}_{0} = a \cdot a = \underline{a^2}$

$\vdots \qquad \vdots$

$h[n] = a\,h[n-1] = a^n$

$\rightarrow h[n] = a^n u[n]$