Chap 5. Discrete-Time Fourier Transform (DTFT)

- Def of DTFT also comes from passing DT sinewave thru a DT LTI system.

$$e^{j\omega_0 n} \longrightarrow \boxed{\begin{matrix}\text{LTI}\\ h[n]\end{matrix}} \longrightarrow y[n] = e^{j\omega_0 n} * h[n]$$

$$= H(\omega_0)\, e^{j\omega_0 n}$$

where $h[n] \overset{\mathcal{F}}{\longleftrightarrow} H(\omega)$

(proof) $y[n] = x[n] * h[n] = \sum_k h[k]\, x[n-k]$

$$= \sum_k h[k]\, e^{j\omega_0 (n-k)} = \underbrace{\left\{ \sum_k h[k]\, e^{-jk\omega_0} \right\}}_{= H(\omega_0)} e^{j\omega_0 n}$$

DTFT: $H(\omega) = \sum_n h[n]\, e^{-j\omega n}$

$$\boxed{\text{DTFT} : \quad X(\omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}}$$

On the next 2 pages, we prove

If $x[n] = x_a(nT_s)$, n: integer and

analog $x_a(t) \overset{\mathcal{F}}{\longleftrightarrow} X_a(\omega)$, $x[n] \overset{DTFT}{\longleftrightarrow} X(\omega)$

then $X(\omega)$ is related to $X_a(\omega)$ as

$$\boxed{\begin{aligned} X(\omega) &= X_s(F_s\omega) \\ \text{where } X_s(\omega) &= F_s \sum_k X_a(\omega - k2\pi F_s) \end{aligned}}$$

, $F_s = \frac{1}{T_s}$: sampling rate (# of samples per sec)

- To derive this relationship, recall that for "sampled" signal, we have

$$x_s(t) = x_a(t)\left(\sum_n \delta(t-nT_s)\right) = \sum_n x_a(t)\,\delta(t-nT_s)$$

$$= \sum_n \underbrace{x_a(nT_s)}_{=x[n]}\,\delta(t-nT_s) = \sum_n x[n]\,\delta(t-nT_s)$$

- Taking the CTFT:

$$X_s(\omega) = \sum_n x[n]\, e^{-j\omega n T_s} = \sum_n x[n]\, e^{-j\left(\frac{\omega}{F_s}\right)n}$$

- Comparing with formula for DTFT obtained from passing DT sinewave thru DT LTI system

$$X(\omega) = \sum_n x[n]\, e^{-j\omega n}$$, we have $X(\omega) = X_s(F_s\omega)$ //

- In our previous derivation for Sampling Theory, rather than bring $x_a(t)$ inside the sum and taking CTFT, we invoked the time-domain product property of FT.

$$x_s(t) = x_a(t)\left(\sum_n \delta(t-nT_s)\right) \xleftrightarrow{\mathcal{F}} \frac{1}{2\pi} X_a(\omega) * \frac{1}{T_s}\sum_k \delta\left(\omega - k\frac{2\pi}{T_s}\right)$$

$$= \underbrace{F_s \sum_k X_a(\omega - k2\pi F_s)}_{\text{where } X_s:} = X_s(\omega)$$

This complete proof

$$\text{DTFT} \quad \boxed{X(\omega) = X_s(F_s\omega)} \quad \text{sampled signal from Analog}$$

$$\Rightarrow X(\omega) = F_s \sum_k X_a(F_s\omega - k2\pi F_s) = F_s \sum_k X_a(F_a(\omega - k2\pi))$$

- Recall fundamental principle

★★★ sample in time domain $\longleftrightarrow$ replications in freq-domain (periodically spaced every integer multiple of $\omega_s = 2\pi F_s$)

- The compression by F_s causes the replications in DTFT to occur at every integer multiple of $\frac{\omega_s}{F_s} = 2\pi$.

Thus, DTFT is periodic with period 2π.

$$X(\omega + \underset{\substack{\uparrow \\ \text{integer}}}{l}\, 2\pi) = \sum_n x[n]\, e^{-j(\omega + l 2\pi)n}$$

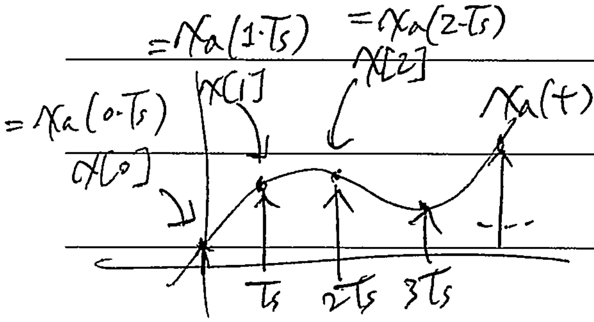
$$= \Big\{ \sum_n x[n]\, e^{-j\omega n} \Big\} \big(\underbrace{e^{-j2\pi}}_{=1} \big)^{ln}$$

$$= \sum_n x[n]\, e^{-j\omega n} = X(\omega)$$

- Further consider sampling a sinewave : $x_a(t) = e^{jW_a t}$ (a: analog)

$$\boxed{x[n]} = x_a(nT_s) = x_a\left(\frac{n}{F_s}\right) = e^{jW_a \frac{n}{F_s}}$$

$$= e^{j\left(\frac{W_a}{F_s}\right)n}$$



- The freq of the resulting DT sinewave is

$$\boxed{\omega_d = \frac{W_a}{F_s}}$$: division by the sampling rate (compression by the " ")

- The DT freq variable may be viewed as a normalized freq variable.

⇒ That is, regardless of the sampling rate, the DTFT is periodic with period 2π

(the replications occur every integer multiple of 2π)

- examine relationship btw DT freq variable (ω_d) and Analog freq variable (ω_a)

$$\omega_d = \frac{\omega_a}{F_s}$$

- If we sample at a rate ω_s ($= 2\pi F_s$), the highest freq we can see is $\frac{\omega_s}{2}$ $\left(= 2\pi \frac{F_s}{2} = \underline{\pi F_s}\right)$

This highest analog freq is mapped to the DT freq.

$$\frac{\pi F_s}{F_s} = \text{“}\pi\text{”}$$

$\Rightarrow$ This is why π is the highest DT freq!

- Any DT freq outside the range $-\pi < \omega < \pi$, you can add or subtract an integer multiple of 2π to put in $-\pi < \omega < \pi$