1. Let A and B be measurable, and show

$$\mu(A \cup B) + \mu(A \cap B) = \mu(A) + \mu(B).$$

2. Show TFAE:

   (a) A is measurable.

   (b) For ever $\epsilon > 0$ there is an open set O such that $A \subseteq O$ and $\mu^*(O - A) < \epsilon$.

   (c) For ever $\epsilon > 0$ there is a closed set F such that $A \supseteq F$ and $\mu^*(A - F) < \epsilon$.

   (d) There is a G_δ set, say G and a set of measure 0, say N such that $A = G - N$.

   (e) There is an F_σ set, say F and a set of measure 0, say M such that $A = F \cup M$.

   (Hint: reduce to the case $A \subseteq [0, 1]$.)

   A G_δ set is a set of the form $\cup_{j=1}^{\infty} O_j$ with O_j open, (G for open in German and δ for product in ???)

   An F_σ set is a set for the form $\cap_{j=1}^{\infty} F_j$ with F_j closed, (F for closed in French and σ for sum in French as well).

3. Show all set of outer measure 0 are measurable.

4. Show A is measurable if and only if for every open set O,

$$\mu^*(O - A) = \mu^*(O) - \mu^*(A).$$

5. Since $\mu(\mathbb{Q}) = 0$, and $\mu = \mu^*$ on the measurable sets, we know that given any $\epsilon > 0$ there is some open set, O such that $\mathbb{Q} \subseteq O$ and $\mu(O) < \epsilon$. Exhibit such an O.

6. (a) Let $\{a_n\}_{\mathbb{N}}$ be a sequence, and notice this means $a : \mathbb{N} \to \mathbb{R}$. Endow $\mathbb{N}$ with the counting measure, i.e. $\mu(\{n\}) = 1$, and inductively $\mu(\{n_1, ..., n_k\}) = k$. What is $\int a(n) d\mu(n)$?

(b) Suppose for each $k \in \mathbb{N}$ we have a sequence $\{a_k(n)\}_n$ such that $a_k(n) \leq a_{k+1}(n)$.Show

$$\lim_k \sum_n a_k(n) = \sum_n \lim_k a_k(n).$$

7. Suppose f_n are measurable, $f_n \to f$ (pointwise), and there is an integrable function g such that $f_n \geq g$ for every n. Show

$$\int f_n \to \int f$$

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Hint: Start with $g = 0$.

8. Let f be measurable, and $\epsilon > 0$. Show there is a continuous g such that $|f - g| < \epsilon$ except on a measurable set, A with $\mu(A) < \epsilon$.