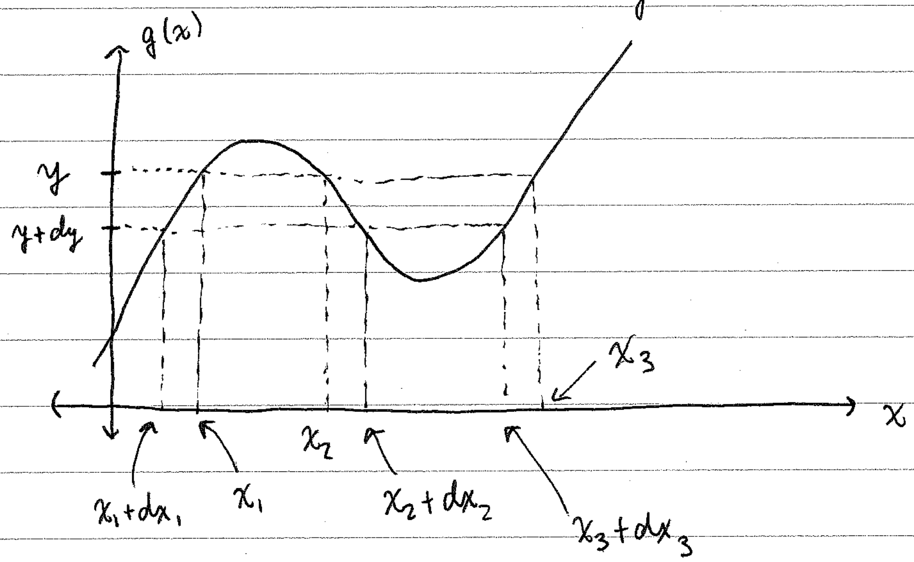
By transitivity,

$$\Rightarrow f_Y(y) = f_X(x) \frac{dx}{dy} \qquad \left|\frac{dy}{dx}\right|^{-1}$$

$$= f_X(x) \left|\frac{dx}{dy}\right| \quad , \quad g'(x) = \frac{dy}{dx} > 0$$

Now consider the following function (N:1)



dx_1, dx_2, dx_3 small

dy small

dy negative

$$Pr(y+dy < Y \le y) = f_Y(y)\,|dy|$$

$$Pr(y+dy < Y \le y) = Pr(\{x_1+dx_1 < X \le x_1\} \cup \{x_2 < X \le x_2+dx_2\} \cup \{x_3+dx_3 < X \le x_3\})$$

$$= f_X(x_1)|dx_1| + f_X(x_2)|dx_2| + f_X(x_3)|dx_3|$$

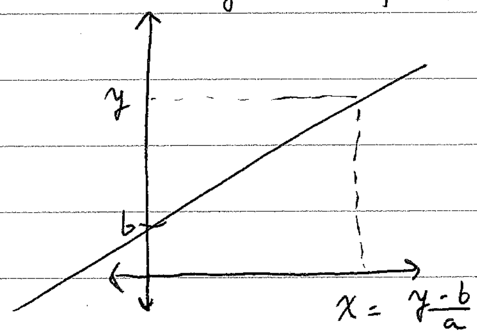
Again, by transitivity

$$f_Y(y) = \sum_k f_X(x) \left|\frac{dx}{dy}\right| \Bigg|_{x=x_k}$$, where x_k are the N solutions to $g(x) = y$

~~Ex:~~ Note: value of $f_Y(y)$ can be assigned arbitrarily at $y = g(x)$ where $\frac{dy}{dx} = 0$

Ex $g(x) = ax + b$, $(a > 0)$

$Y = g(X)$, X is a continuous r.v.



Density Method

$g(x)$ is 1:1

$$f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right|, \quad y \in \mathbb{R}$$

$$y = ax + b \iff x = \frac{y-b}{a}$$

$$\frac{dy}{dx} = a \Rightarrow \left|\frac{dx}{dy}\right| = \frac{1}{a}$$

$$f_Y(y) = f_X\left(\frac{y-b}{a}\right)\frac{1}{a}, \quad y \in \mathbb{R}$$

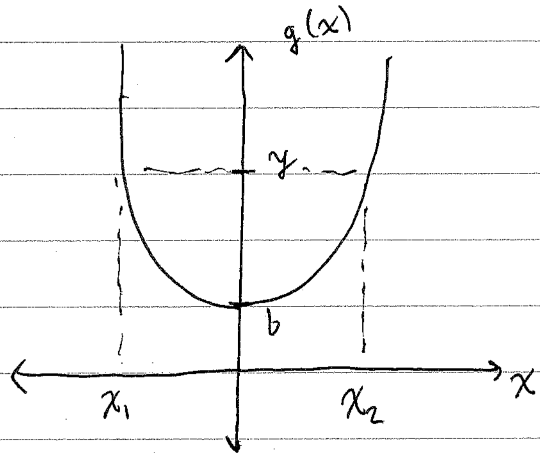
If $f_X(x) = e^{-x}u(x)$

$$\Rightarrow f_Y(y) = \frac{1}{a}\exp\left(-\left(\frac{y-b}{a}\right)\right)u\left(\frac{y-b}{a}\right)$$

$$= \frac{1}{a}\exp\left(-\frac{y-b}{a}\right)u(y-b)$$

__Ex__: $g(x) = ax^2 + b$, $(a > 0)$

$Y = g(X)$, X is a continuous r.v.



$$x_1 = -\sqrt{\frac{y-b}{a}}$$

$$x_2 = \sqrt{\frac{y-b}{a}}$$

$g(x)$ is not 1:1

$g(x)$ is 2:1 for $y > b$

Density Method

$\cancel{f_Y(y)}$

$$f_Y(y) = f_X(x_1)\left|\frac{dx_1}{dy}\right| + f_X(x_2)\left|\frac{dx_2}{dy}\right| \quad , \quad y > b$$

$$= 0 \quad , \quad y \le b$$

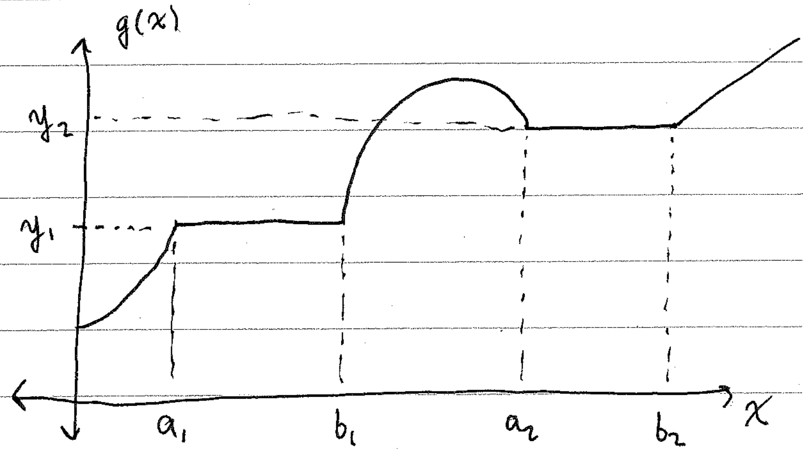
$$y = ax^2 + b \Rightarrow x_1 = -\sqrt{\frac{y-b}{a}} \quad , \quad x_2 = \sqrt{\frac{y-b}{a}}$$

$$\left|\frac{dx_1}{dy}\right| = \frac{1}{2a|x_1|} = \frac{1}{2a\sqrt{a(y-b)}} = \left|\frac{dx_2}{dy}\right|$$

$$f_Y(y) = \left(f_X\left(-\sqrt{\frac{y-b}{a}}\right) + f_X\left(\sqrt{\frac{y-b}{a}}\right)\right)\frac{1}{2\sqrt{a(y-b)}} \quad , \quad y > b$$

$$= 0 \quad , \quad y \le b$$

Functions of continuous r.v.s may result in mixed r.v.s.



$Pr(Y=y_1)$ and $Pr(Y=y_2)$ may not be 0.

The distribution method can still be used as derived before.

The density method needs an adjustment.

Remember if Y is a mixed r.v., we can write

$$F_Y(y) = p F_1(y) + (1-p) F_2(y), \quad 0 \le p \le 1$$

$$\Rightarrow \quad f_Y(y) = p f_1(y) + (1-p) f_2(y)$$

where $f_1(y)$ is a continuous pdf and $f_2(y)$ is a discrete pdf

Will just write

$$f_Y(y) = f_1(y) + f_2(y)$$

First find $f_1(y)$ as before for values of y
for $g(x)$ is N:1

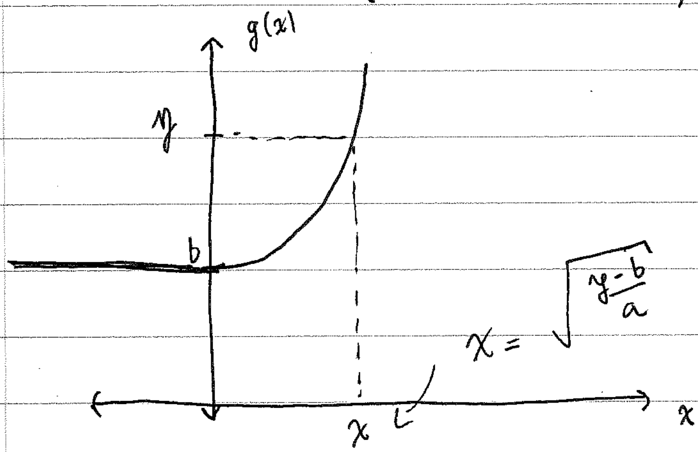
$$f_1(y) = \sum_{K=1}^{N} f_X(x_K) \left| \frac{dx_K}{dy} \right| , \quad y = g(x_1) = \ldots = g(x_N)$$

Then find $f_2(y)$ for values $y_j = g(x)$ for $x \in [a_j, b_j]$

$$f_2(y) = \sum_{j=1}^{M} P_r(Y = y_j)\, \delta(y - y_j)$$

$$= \sum_{j=1}^{M} \left(\int_{a_j}^{b_j} f_X(x)\, dx \right) \delta(y - y_j)$$

Ex $\quad g(x) = \begin{cases} ax^2 + b, & x \geq 0 \\ b, & x < 0 \end{cases} \qquad (a > 0)$



Density Method

$$f_Y(y) = f_1(y) + f_2(y)$$

$$f_1(y) = f_X(x)\left|\frac{dx}{dy}\right| \quad , \quad y > b \quad (x > 0)$$
$$= 0 \quad , \quad \text{else}$$

$$f_1(y) = f_X\left(\sqrt{\frac{y-b}{a}}\right)\frac{1}{2\sqrt{a(y-b)}} \quad , \quad y > b$$
$$= 0 \quad , \quad \text{else}$$

$$f_2(y) = \Pr(Y=b)\,\delta(y-b)$$
$$= \Pr(X \le 0)\,\delta(y-b)$$
$$= \left(\int_{-\infty}^{0} f_X(x)\,dx\right)\delta(y-b)$$

If $f_X(x) = e^{-x}u(x) \Rightarrow \Pr(X \le 0) = 0$

Distribution Method

$$F_Y(y) = \int_{x:\,g(x)\le y} f_X(x)\,dx$$
$$= \int_{-\infty}^{\sqrt{\frac{y-b}{a}}} f_X(x)\,dx \quad , \quad y \ge b$$
$$= \int_{0}^{\sqrt{\frac{y-b}{a}}} f_X(x)\,dx + \int_{-\infty}^{0} f_X(x)\,dx \quad , \quad y \ge b$$

$$\Rightarrow f_Y(y) = \frac{d}{dy}F_Y(y)$$
$$= f_X\left(\sqrt{\frac{y-b}{a}}\right)\frac{1}{2\sqrt{a(y-b)}} + \left(\int_{-\infty}^{0} f_X(x)\,dx\right)\delta(y-b) \quad , \quad y \ge b$$
$$= 0 \quad , \quad y < b$$