8. Let $A, B \subset \mathbb{R}$, $A \neq \emptyset$ $B \neq \emptyset$, A & B are sets of positive reals & bounded above.

Set $AB = \{ab : a \in A, b \in B\}$. We show $\sup(AB) = \sup(A)\sup(B)$

pf: Lets set $\alpha = \sup(A) < \infty$ $\beta = \sup(B) < \infty$ (since A & B are bdd above).

$A \neq \emptyset$ & $B \neq \emptyset \Rightarrow \exists\, a \in A,\ b \in B$.

$\Rightarrow 0 < a \leq \alpha \qquad 0 < b \leq \beta$ so α, β positive.

Next $\forall\, ab \in AB$ $(a \in A \ b \in B)$

$a \leq \alpha$ $b \leq \beta$ so $ab \leq \alpha\beta$.

$\Rightarrow \alpha\beta$ an upper bound for AB

$\Rightarrow \sup(AB) \leq \alpha\beta$.

Next we show $\sup(AB) \geq \alpha\beta$.

Let $\varepsilon > 0$ be given. Set $\eta = \varepsilon/(\alpha+\beta)$

Choosing ε smaller we may assume $\alpha - \eta \geq 0$ & $\beta - \eta \geq 0$.

$\left.\begin{array}{ll} \exists\, a \in A \ni & \alpha - \eta < a \\ \exists\, b \in B \ni & \beta - \eta < b \end{array}\right\}$ definition of $\sup(A)$ & $\sup(B)$.

$\Rightarrow \quad (\alpha - \eta)(\beta - \eta) < ab$ $\longleftrightarrow$ (Notice here we used that $0 < \alpha - \eta$ & $0 < \beta - \eta$ to preserve our inequality when we multiplied.)

$$\begin{aligned}\Rightarrow \alpha\beta - \varepsilon &= \alpha\beta - \eta(\alpha+\beta) \\ &\leq \alpha\beta - \eta(\alpha+\beta-\eta) \\ &= (\alpha-\eta)(\beta-\eta) < ab \leq \sup(AB)\end{aligned}$$

So $\alpha\beta - \varepsilon \leq \sup(AB)$ but ε arbitrary. $\blacksquare$