Name: ______________________________

EE-602
Exam I
February 2, 2006

140 Point Exam

INSTRUCTIONS

This is a closed book, closed notes exam. No calculator is permitted. Work patiently, efficiently, and in an organized manner clearly identifying the steps you have taken to solve each problem. Your grade for each problem depends on the COMPLETENESS, ORGANIZATION and CLARITY of your work as well as the resulting answer. All students are expected to abide by the usual ethical standards of the university, i.e., your answers must reflect only your own knowledge and reasoning ability.

There are a total of 14 pages.

Good luck.

PART 1. SHORT TAKES (45 POINTS)

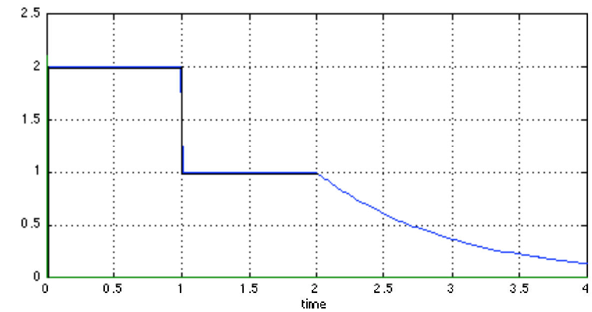
1. (8 pts) Suppose a multi-input multi-output system is characterized by the set of differential equations

$$\ddot{y}_1(t) + a_1\ddot{y}_2(t) + a_3\left[y_1(t) - y_2(t)\right] = \sin^2(t-1)u_1(t) - a_4\sin\left[tu_2(t)\right]$$
$$\ddot{y}_2(1-t) + b_1\dot{y}_1(t) + b_2 y_2(t) = u_2(t)$$

In the above set of system equations:
(a) What terms if any make the system nonlinear?
(b) What terms if any make the system time varying?

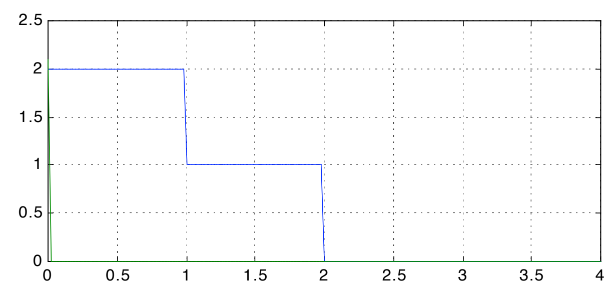
(c) What terms if any make the system causal or not causal?
(d) What terms if any make the system lumped or distributed?

2. (5 pts) It is known that a non-causal system is linear and time invariant. The response of this system to the input $u(t) = 1^+(t)$ is $y(t) = (2 - e^{-t})1^+(t)$. Determine as much of the response $\hat{y}(t)$ as possible to the input



$\hat{y}(t)$ = ______________________________

3. (3 pts) Repeat 2 for the input below.



$\hat{y}(t)$ = ______________________________

4. (13 pts) (a) (2 pts) State the definition of a linear system $L : U \rightarrow Y$.
(b) (3 pts) Prove that $L(0-\mathit{function}) = 0-\mathit{function}$

(c) (8 pts) Prove that a system $N: U \rightarrow Y$ is linear if and only if $N\left[\alpha_1 u_1(t) + u_2(t)\right] = \alpha_1 N\left[u_1(t)\right] + N\left[u_2(t)\right]$.

5. (16 pts) (a) (3 pts) State the definition of zero-input zero-output causality.
(b) (3 pts) State the definition of causality for a general (nonlinear) system.
(c) (10 pts) Prove that for a linear system, the (correct) definitions of (a) and (b) are equivalent.

PART 2. REAL PROBLEMS

1. (20 pts) (a) (9 pts) Prove that if $A_1 A_2 = A_2 A_1$ and $A = A_1 + A_2$, then $e^{At} = e^{A_1 t} e^{A_2 t}$.
(b) (11 pts) Use the result of part (a) and another result (or two) from the class notes to construct the solution, $x(t)$ to $\dot{x} = Ax$ (to the maximum extent possible), when $x(t_0) = x_0$ and

$$A = T \left[\begin{array}{ccc|c} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 0 \\ \hline 0 & 0 & 0 & -1 \end{array}\right] T^{-1}$$

2. (20 pts) (a) (10 pts) Many mechanical systems have a matrix differential equation model of the form

$$M\ddot{q} + D\dot{q} + Kq = Bu$$

where $q(t) \in R^n$ represents a vector of generalized displacements, M is a $n \times n$ matrix of masses, D is a $n \times n$ matrix of damping coefficients, and K is a $n \times n$ stiffness matrix. Define a state vector of the form $x = \begin{bmatrix} q \\ \dot{q} \end{bmatrix}$ and construct a homogeneous state model for the generalized mechanical system. What conditions on M are sufficient for the existence of the state model?
(b) (10 pts) The equations of motion of a two-mass two-spring system on a horizontal frictionless surface are given by

$$m_1 \ddot{x}_1 = -k_1 x_1 + k_2 (x_2 - x_1) + b_1 u_1$$
$$m_2 \ddot{x}_2 = -k_2 (x_2 - x_1) + b_2 u_2$$

Write these equations in the form of a general driven mechanical system and use the result of part (a) to write down the corresponding state model.

3. (20 pts) The nonlinear state model of the vertical ascent of a rocket above the earth's surface is given by the equation below where $x_1 = r$ is the distance above the earth's surface and R is the distance from the center of the earth to the surface, on average. The quantity m is the mass of the rocket and g is the acceleration due to gravity.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -g\left(\dfrac{R}{x_1 + R}\right)^2 + \dfrac{1}{m} T(t) \end{bmatrix}.$$

(a) If the thrust T(t) = T* is a positive constant, find the constant equilibrium solutions.
(b) Find the linearized state dynamics about the constant equilibrium solutions. Designate the constant equilibrium solutions by T^*, x_1^*, and x_2^*. Use this notation in your answer!

4. (23 pts) Consider the state dynamics

$$\dot{x}(t) = Ax(t) + Bu(t),\ x(2) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

where $A = aM$ where the matrix M satisfies $M^2 = M$ and $B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

(a) (5 pts) Show by induction that A has the property that $A^k = a^k M$.
(b) **(8 pts)** Derive the formula $e^{At} = I + M(e^{at} - 1)$.
(c) (10 pts) If $M = \begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix}$ and $u(t) = e^{at} 1^+(t)$ find x(0).

5. (12 pts) Identity DC-gain for such a state model

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t)$$

means that for any constant m-vector u_0, there exists a constant n-vector x_0 such that

$$0 = Ax_0 + Bu_0$$
$$y_0 = Cx_0 = u_0$$

ASSUME: if there exists a m×n matrix K such that $(A+BK)^{-1}$ exists, then $C(A+BK)^{-1}B$ is invertible.

PROVE: if there exists a m×n matrix K such that $(A+BK)^{-1}$ exists, then a m×m matrix N exists for which the state model

$$\dot{x}(t) = \left(A+BK\right)x(t) + BNu(t)$$
$$y(t) = Cx(t)$$

has identity DC-gain.

Name: BRIEF SOLUTIONS

EE-602
Exam I
September 15, 2005

130 Point Exam

INSTRUCTIONS

This is a closed book, closed notes exam. No calculator is permitted. Work patiently, efficiently, and in an organized manner clearly identifying the steps you have taken to solve each problem. Your grade for each problem depends on the COMPLETENESS, ORGANIZATION and CLARITY of your work as well as the resulting answer. All students are expected to abide by the usual ethical standards of the university, i.e., your answers must reflect only your own knowledge and reasoning ability.

There are a total of 14 pages , some T-F and FIB, & 6 problems.

Good luck.

PART 1. (25 PTS) TRUE-FALSE AND FILL-IN-THE-BLANK. Write out the word TRUE or FALSE or lose 3 pts. In answering the True of False, 2 pts for correct answer, 0 pts for no answer, and –1 pt for incorrect answer. Avoid guessing.

1. (4 pts) A linear time invariant lumped system has response $y_1(t)$ to the the admissible input $u_1(t)$ and the response $y_2(t)$ to the admissible input $u_2(t)$. What is the response to $\dot{u}_1(t-T) - \beta u_2(t+T)$?

$\dot{y}_1(t-T) - \beta y_2(t+T)$

2. (T/F) (2 pts) A system is represented by the convolution integral $y(t) = \int_{-\infty}^{+\infty} H(t,q)u(q)dq$. If $H(t,q) = \left[tq1^+(t) \quad t-q\right]$, the system is causal. **FALSE** Note the $H(1,2) = \left[2 \quad -1\right] \neq 0$.

3. (T/F) (2 pts) For the linear differential equation $\ddot{y}(t) + a_1\dot{y}(t) + a_2 ty(t) = u(t)$, the response to $\beta u(t-T)$ is $\beta y(t-T)$, $T > 0$. **FALSE** (time varying diff eq)

4. (F-I-Blank) (6 pts) A causal time invariant lumped system has the response y(t) to the admissible input $u(t) = \beta\left(1^+(t) - 1^+(t-T)\right)$ for $T > 0$. Then the response to the new input $\hat{u}(t) = \beta 1^+(t-2T)$ is

(specify all time intervals): $\hat{y}(t) = \begin{cases} y(t-2T) & t \le 3T \\ unknown & t > 3T \end{cases}$

5. (6 pts) Suppose a multi-input multi-output system is characterized by the set of differential equations

$$\ddot{y}_1(t) + a_1\ddot{y}_2(t) - a_2\left[y_1(0.25-t) + y_2(t)\right] = u_1(t)y_1(t) - e^{-t}u_2(t)$$
$$\ddot{y}_2(t) + t\dot{y}_1(t) - y_1(t)y_2(t) = u_2(t)$$

In the above set of system equations:

(a) What terms if any make the system nonlinear? $u_1(t)y_1(t)$ and $y_1(t)y_2(t)$

(b) What terms if any make the system time varying? $e^{-t}u_2(t)$ and $t\dot{y}_1(t)$

(c) What terms if any make the system causal or not causal? $y_1(0.25-t)$

(d) What terms if any make the system lumped or distributed? $y_1(0.25-t)$

6. (5 pts) Suppose a 2nd order linear lumped time-invariant causal state model has scalar output $y(t)$. If the zero-input response to $x(0) = \begin{bmatrix} 1 & 0 \end{bmatrix}^T$ is $y(t) = t1^+(t)$ and the zero-input system response to $x(0) = \begin{bmatrix} 0 & -1 \end{bmatrix}^T$ is $y(t) = te^{-t}1^+(t)$. Find the zero-input system response to the initial condition

$x(T) = \begin{bmatrix} \alpha & \beta \end{bmatrix}^T$, $T > 0$. $y_{new}(t) = \alpha(t-T)1^+(t-T) - \beta(t-T)e^{-(t-T)}1^+(t-T)$

PART 2. PROBLEMS

1. (12 pts) For this problem you may not solve the equations or use a solution form as a basis for proving linearity.
(a) (2 pts) State the definition of a linear system denoted by the operator L as was done in class.
(b) (10 pts) Prove that the zero-input state-response of the state dynamics

$$\dot{x}(t) = A(t)x(t) + B(t)u(t)$$

is linear in the initial condition, $x(t_0)$, i.e., let $x^1(t_0)$ and $x^2(t_0)$ be two initial conditions having state responses, $x^1(t)$ and $x^2(t)$ respectively for $t \ge t_0$. Prove that an arbitrary linear combination of the initial states yields the same linear combination of the zero-input state-responses.

(a) See notes

(b) For zero-input state response, the state dynamics $\dot{x}(t) = A(t)x(t) + B(t)u(t)$ reduces to $\dot{x}(t) = A(t)x(t)$. Let $x^1(t_0)$ and $x^2(t_0)$ be two initial conditions having zero-input state responses $x^1(t)$ and $x^2(t)$ respectively for $t \ge t_0$. For arbitrary ε_1 and ε_2,

$$\begin{aligned}\frac{d(\varepsilon_1 x^1(t) + \varepsilon_2 x^2(t))}{dt} &= \varepsilon_1\frac{dx^1(t)}{dt} + \varepsilon_2\frac{dx^2(t)}{dt}\\ &= \varepsilon_1 A(t)x^1(t) + \varepsilon_2 A(t)x^2(t)\\ &= A(t)(\varepsilon_1 x^1(t) + \varepsilon_2 x^2(t))\end{aligned}$$

Hence, $\varepsilon_1 x^1(t) + \varepsilon_2 x^2(t)$ is a solution of the differential equation $\dot{x}(t) = A(t)x(t)$. Let $x^3(t_0) = \varepsilon_1 x^1(t_0) + \varepsilon_2 x^2(t_0)$ be an initial condition having zero-input state response $x^3(t)$ for $t \ge t_0$. Note that $x^3(t)$ and $\varepsilon_1 x^1(t) + \varepsilon_2 x^2(t)$ are equal at $t = t_0$ (i.e., same initial condition), and both solutions satisfy the same differential equation, $\dot{x}(t) = A(t)x(t)$ for $t \ge t_0$. It follows from the uniqueness theorem that $x^3(t) = \varepsilon_1 x^1(t) + \varepsilon_2 x^2(t)$ for $t \ge t_0$. Thus, an arbitrary linear combination of the initial states yields the same linear combination of the zero-input state responses.

2. (25 pts) The state model of a software test process is given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \kappa_1 w_f & \dfrac{\kappa_2}{\gamma} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad \text{(Equation 1)}$$

where x_1 is the number of errors remaining in a software product and x_2 is the velocity with which the workforce, denoted w_f ($\neq 0$), finds and fixes the errors. γ is the quality of the workforce and is a number between zero and one. Both $w_f = u_1$ and $\gamma = u_2$ are inputs to the system and are functions of time in general.
(a) (4 pts) Is the system: causal or noncausal, time invariant or not time invariant, linear or not linear, lumped or distributed?
(b) (4 pts) Determine the constant equilibrium solutions x_1^* and x_2^* given constant inputs of $u_1^* = w_f^* \neq 0$ and $u_2^* = \gamma^* \neq 0$.

(c) (6 pts) Now suppose that $\kappa_1 = -0.4$, $u_1^* = w_f^* = 5$ (five testers), $u_2^* = \gamma^* = 1$ (highest quality), and $\kappa_3 = -3$. With these parameter values,

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}1 & 1\\ -1 & -2\end{bmatrix}\begin{bmatrix}-1 & 0\\ 0 & -2\end{bmatrix}\begin{bmatrix}2 & 1\\ -1 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

Find the non-constant nominal solution vector $x^*(t)=\begin{bmatrix}x_1^* & x_2^*\end{bmatrix}^T$ assuming $x_1(0)=R_0$ is the total number of errors present in the software product and $x_2(0)=0$, i.e., the initial speed with which the testers find errors is zero.

(d) (11 pts) Linearize the system (Equation 1) about the nominal solutions NOT the constant equilibrium solutions of part (b). Write down the complete linearized perturbation state model of the system in terms of the variables x* and u*, NOT their values as computed in (c).

Solution. (a) causal, time invariant, not linear, lumped.

(b)

$$\begin{bmatrix}0\\ 0\end{bmatrix}=\begin{bmatrix}0 & 1\\ \kappa_1 w_f^* & \dfrac{\kappa_2}{\gamma^*}\end{bmatrix}\begin{bmatrix}x_1^*\\ x_2^*\end{bmatrix}\Rightarrow\begin{bmatrix}x_1^*\\ x_2^*\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

(c)

$$\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix}=\begin{bmatrix}1 & 1\\ -1 & -2\end{bmatrix}\begin{bmatrix}e^{-t} & 0\\ 0 & e^{-2t}\end{bmatrix}\begin{bmatrix}2 & 1\\ -1 & -1\end{bmatrix}\begin{bmatrix}x_1(0)\\ x_2(0)\end{bmatrix}=\begin{bmatrix}1 & 1\\ -1 & -2\end{bmatrix}\begin{bmatrix}e^{-t} & 0\\ 0 & e^{-2t}\end{bmatrix}\begin{bmatrix}2 & 1\\ -1 & -1\end{bmatrix}\begin{bmatrix}R_0\\ 0\end{bmatrix}$$

(d)

$$A(t)=\frac{\partial}{\partial x}\left(\begin{bmatrix}0 & 1\\ \kappa_1 w_f & \dfrac{\kappa_2}{\gamma}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}\right)_{x^*,u^*}=\begin{bmatrix}0 & 1\\ \kappa_1 w_f^* & \dfrac{\kappa_2}{\gamma^*}\end{bmatrix}$$

$$B(t)=\frac{\partial}{\partial u}\left(\begin{bmatrix}0 & 1\\ \kappa_1 w_f & \dfrac{\kappa_2}{\gamma}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}\right)_{x^*,u^*}=\begin{bmatrix}0 & 0\\ \kappa_1 x_1^* & \dfrac{-\kappa_2 x_2^*}{\left(\gamma^*\right)^2}\end{bmatrix}$$

$$\begin{bmatrix}\dot{\Delta x}_1\\ \dot{\Delta x}_2\end{bmatrix}=A(t)\begin{bmatrix}\Delta x_1\\ \Delta x_2\end{bmatrix}+B(t)\begin{bmatrix}\Delta u_1\\ \Delta u_2\end{bmatrix}$$

3. (15 pts) The equations of an electrostatic microphone[1] are

$$R\frac{dQ}{dt}+\frac{Q}{B}h-E=0$$
$$m\frac{d^2h}{dt^2}+\mu\frac{dh}{dt}+k(h-L)+\frac{Q^2}{2B}=F$$

where h is the displacement of the diaphragm from the backplate, C is the capacitance between the diaphragm and the back-plate, E is a bias voltage in the circuit, Q is the charge on the internal capacitor, F is the total force exerted on the diaphragm by one's voice and is assumed to be uniform. The various constants of the model are as follows: m is the combined mass of the diaphragm and the air accelerated by the diaphragm, μ is the damping constant, k is the elastic constant of the microphone, L denotes the diaphragm position when no force acts on the diaphragm, R is a resistance, and $B\cong hC$ is an experimentally determined constant.

Choose an appropriate set of state variables denoted by $x_i,\ i=1,\cdots$?? (you must determine the proper number of state variables) and then construct a (nonlinear) state model of the electrostatic microphone..

Solution. Let $x_1=Q$, $x_2=h$, and $x_3=\dot{h}$. Then

$$\begin{aligned}&R\dot{x}_1+\frac{1}{B}x_1x_2-E=0\\ &m\dot{x}_3+\mu x_3+k(x_2-L)+\frac{1}{2B}x_1^2=F\end{aligned}\quad\text{implies}\quad\begin{aligned}&\dot{x}_1=-\frac{1}{RB}x_1x_2+\frac{1}{R}E\\ &\dot{x}_3=-\frac{1}{2Bm}x_1^2-\frac{k}{m}(x_2-L)-\frac{\mu}{m}x_3+\frac{1}{m}F\end{aligned}$$

Therefore the nonlinear state model is given by

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix}=\begin{bmatrix}-\dfrac{1}{RB}x_1x_2+\dfrac{1}{R}E\\ x_3\\ -\dfrac{1}{2Bm}x_1^2-\dfrac{k}{m}(x_2-L)-\dfrac{\mu}{m}x_3+\dfrac{1}{m}F\end{bmatrix}$$

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4. (32 pts) Consider the state dynamics $\dot{x} = Ax$ where $A = \left[\begin{array}{c|c} M_1 & 0 \\ \hline M_2 & M_3 \end{array}\right]$ is a block diagonal matrix

with $M_1 = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$, $M_3 = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$, $M_2 = \begin{bmatrix} 0 & \lambda \\ 0 & \lambda \end{bmatrix}$

(a) State simplified conditions on the M_i under which $\left[\begin{array}{c|c} M_1 & 0 \\ \hline 0 & M_3 \end{array}\right]$ and $\left[\begin{array}{c|c} 0 & 0 \\ \hline M_2 & 0 \end{array}\right]$ commute.

(b) Find expressions for $e^{M_1 t}$, $e^{M_3 t}$.

(c) Find a SIMPLIFIED expression for e^{At}. Gotta do the mults.

(d) If $x(1) = \begin{bmatrix} 0 & 1 & 0 & 1 \end{bmatrix}^T$ find $x(0)$.

Solution. (a) (6 pts) $M_3 M_2 = M_2 M_1$.

(b) (6 pts) $e^{M_1 t} = e^{\lambda t}\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$, (4 pts) $e^{M_3 t} = e^{\lambda t} I$.

(c) (16 pts) Observe that $M_3 M_2 = M_2 M_1$ (2 pts). Therefore

$A = \left[\begin{array}{c|c} M_1 & 0 \\ \hline M_2 & M_3 \end{array}\right] = \left[\begin{array}{c|c} M_1 & 0 \\ \hline 0 & M_3 \end{array}\right] + \left[\begin{array}{c|c} 0 & 0 \\ \hline M_2 & 0 \end{array}\right]$ is decomposed as a sum of commuting matrices. Hence

$$e^{At} = \left[\begin{array}{c|c} e^{M_1 t} & 0 \\ \hline 0 & e^{M_3 t} \end{array}\right]\left[\begin{array}{c|c} I & 0 \\ \hline M_2 t & I \end{array}\right] = e^{\lambda t}\begin{bmatrix} 1 & t & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \lambda t & 1 & 0 \\ 0 & \lambda t & 0 & 1 \end{bmatrix} \text{ (14 pts)}$$

(d) (4 pts) $e^{-\lambda}\begin{bmatrix} -1 & 1 & -\lambda & -\lambda+1 \end{bmatrix}^T$

5. (10 pts) Since e^{At} satisfies the matrix differential equation $\dot{M} = AM$, $M(0) = I$, use the uniqueness theorem to show that if $A = TDT^{-1}$, then $e^{At} = Te^{Dt}T^{-1}$.

Solutiion. We know from class result that $\frac{d}{dt}e^{At} = Ae^{At}$, $e^{At}\big|_{t=0} = I$. Observe that

$$\frac{d}{dt}\left(Te^{Dt}T^{-1}\right) = T\frac{d}{dt}\left(e^{Dt}\right)T^{-1} = TDe^{Dt}T^{-1} = TDT^{-1}Te^{Dt}T^{-1}$$

$$= A\left(Te^{Dt}T^{-1}\right),\ Te^{Dt}T^{-1}\Big|_{t=0} = TT^{-1} = I$$

By the uniqueness theorem, $e^{At} = Te^{Dt}T^{-1}$.

6. (11 pts) The matrix A is **idempotent** if $A^2 = A$. Compute a simple expression for e^{At} when A is idempotent.

Solution. $e^{At} = \sum_{k=0}^{\infty} A^k \frac{t^k}{k!} = I + \sum_{k=1}^{\infty} A\frac{t^k}{k!} = I + A\left[\left(\sum_{k=0}^{\infty}\frac{t^k}{k!}\right) - 1\right] = I + (e^t - 1)A$

Check: $I + (e^t - 1)A\big|_{t=0} = I$ and $\frac{d}{dt}\left(I + (e^t - 1)A\right) = e^t A$. But,

$$A\left(I + (e^t - 1)A\right) = A + (e^t - 1)A^2 = A + (e^t - 1)A = e^t A$$

Thus our expression is correct.

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Name: ______________________________

EE-602
Exam I
September 21, 2004

132 Point Exam

INSTRUCTIONS

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There are a total of 12 pages , 6 problems.

Good luck.

PART 1. (30 PTS) TRUE-FALSE AND FILL-IN-THE-BLANK. Write out the word TRUE or FALSE or lose 3 pts. In answering the True of False, 2 pts for correct answer, 0 pts for no answer, and –1 pt for incorrect answer. Avoid guessing.

1. (3 pts) A linear time invariant system has response $y_1(t)$ to the input $u_1(t)$ and the response $y_2(t)$ to the input $u_2(t)$. What is the response to $u_1(t-T)-\beta\dot{u}_2(t)$. $y_1(t-T)-\beta\dot{y}_2(t)$

2. (T/F) (2 pts) A system is represented by the convolution integral $y(t)=\int_{-\infty}^{+\infty} h(t-q)u(q)dq$. If $h(t)=e^t 1^+(-t)$, the system is causal. **FALSE**

3. (T/F) (2 pts) For the linear differential equation $\ddot{y}(t)+a_1\dot{y}(t)+a_2 ty(t)=u(t)$, the response to $\frac{du(t)}{dt}$ is $\frac{dy(t)}{dt}$. **FALSE**

4. **(T/F) (8 pts)** A state model is given by the equation

$$\dot{x}(t)=A_{v(t)}x(t)+Bu(t)$$
$$y(t)=Cx(t)$$

where $v(t)\in\{0,1\}$ is a given function of time taking only the values "0" or "1" so that at each t $A_{v(t)}\in\{A_0,A_1\}$ where A_0 and A_1 are two nxn constant matrices.

(a) The system model is linear. **TRUE**

(b) The system is time invariant. **FALSE**

(c) The system is lumped. **TRUE**

(d) The system is causal. **TRUE**

5. (T/F) (2 pts) Suppose $A=A_1+A_2$ and A_1 and A_2 commute. Then $e^{At}=e^{A_2t}e^{A_1t}$. **TRUE**

6. (T/F) (2 pts) $T(t)=\begin{bmatrix} t & 1-t \\ t & t-1 \end{bmatrix}$ is a valid time varying state transformation matrix. **FALSE**

7. (8 pts) For the state model

$$\dot{x}(t)=\begin{bmatrix} 0 & 0 \\ a & 0 \end{bmatrix}x(t)+Bu(t)$$
$$y(t)=\begin{bmatrix} 1 & 0 \end{bmatrix}x(t)$$

(a) (2 pts) The natural frequencies of the system are: **"0" and "0" or "0" twice**

(b) (3 pts) $e^{At}=\begin{bmatrix} 1 & 0 \\ at & 1 \end{bmatrix}$

(c) (3 pts) The zero-input system response to the initial condition $x(1)=\begin{bmatrix} 1 & 0 \end{bmatrix}^T$ is:

$C\begin{bmatrix} 1 \\ a(t-1) \end{bmatrix}=1$ for t ≥ 1. (I took the zero-input state response.)

8. (3 pts) If $A=\begin{bmatrix} 1 & 0 \\ a & 1 \end{bmatrix}$ then $e^{At}=e^t\begin{bmatrix} 1 & 0 \\ at & 1 \end{bmatrix}$

PART 2. PROBLEMS

1. (12 pts) A mechanical system is described by the coupled linear matrix differential equation

$$M_1\begin{bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \end{bmatrix}+M_2\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix}+M_3\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}=\begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$$

where F_1 and F_2 are input forces, each matrix M_i is 2×2, and M_1^{-1} exists. In writing a state model, state variables are chosen as $x_1 = y_1$, $x_2 = y_2$, $x_3 = \dot{y}_1$, and $x_4 = \dot{y}_2$. Compute A_{21}, A_{22}, and B_2 in the matrix below in terms of the M_i-matrices.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right] \begin{bmatrix} x_1 \\ x_2 \\ \hline x_3 \\ x_4 \end{bmatrix} + \left[\begin{array}{c} B_1 \\ \hline B_2 \end{array}\right] \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$$

SOLUTION 1: Substituting

$$\begin{bmatrix} \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = M_1^{-1} \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} - M_1^{-1} M_2 \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} - M_1^{-1} M_3 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Hence

$$B_2 = M_1^{-1},\ A_{21} = -M_1^{-1} M_3,\ A_{22} = -M_1^{-1} M_2$$

2. (20 pts) The predator-prey model of a biological system with the Bush administration adding quail to the host population in which hunting is open to people with Lugars is:

$$\frac{dH}{dt} = a_1 H - c_1 PH + d_1 u$$
$$\frac{dP}{dt} = -a_2 P + c_2 PH$$

where (i) a_1, a_2, c_1, c_2, and d_1 are all non-zero positive values unequal to each other, (ii) a_1H is the uninhibited host's growth rate, (iii) $-c_1PH$ represents the decrease in growth rate due to the presence of parasites, (iv) $-a_2P$ represents the natural death rate of the parasites, and (v) the product term, c_2HP, models the dependence of the growth rate of the parasites upon the existence of a host.
(a) Determine nonzero constant equilibrium solutions H* ($\neq 0$) and P* ($\neq 0$) given a constant feed rate u* ($\neq 0$).
(b) Linearize the system about these nonzero constant equilibrium solutions. Specify the linearized perturbation state model of the system in terms of the variables P* and H*, NOT the values computed in (a).

SOLUTION 2. **(a)** Consider $0 = -a_2 P^* + c_2 P^* H^* = P^*(c_2 H^* - a_2) \Rightarrow H^* = \frac{a_2}{c_2}$. Hence

$0 = a_1 H^* - c_1 P^* H^* + d_1 u^* \Rightarrow P^* = \frac{a_1}{c_1} + \frac{d_1}{c_1 H^*} u^* = \frac{a_1}{c_1} + \frac{d_1 c_2}{a_2 c_1} u^*$ since H* $\neq 0$.

(b) $A = \begin{bmatrix} a_1 - c_1 P^* & -c_1 H^* \\ c_2 P^* & c_2 H^* - a_2 \end{bmatrix}$, $B = \begin{bmatrix} d_1 \\ 0 \end{bmatrix}$, $\Delta x = \begin{bmatrix} \Delta H \\ \Delta P \end{bmatrix}$, and $\Delta \dot{x} = A\Delta x + B\Delta u$.

3. (14 pts) Suppose A = T D T^{-1} for an appropriate nonsingular matrix T. (Here D is often diagonal, but can be any matrix.)
(a) (6 pts) Show by induction that $A^k = T D^k T^{-1}$.
(b) (8 pts) Use (a) and the definition of the Taylor series to prove Theorem 3.4, i.e., $e^{At} = Te^{Dt}T^{-1}$.

SOLUTION 3. (a) By definition, $A = TDT^{-1}$. Assume $A^{k-1} = TD^{k-1}T^{-1}$. Observe that

$$A^k = AA^{k-1} = TDT^{-1} \times TD^{k-1}T^{-1} = TD^kT^{-1}$$

(b) Using the result of (a)

$$e^{At} = \sum_{k=0}^{\infty} (TDT^{-1})^k \frac{t^k}{k!} = \sum_{k=0}^{\infty} TD^kT^{-1} \frac{t^k}{k!} = T\left(\sum_{k=0}^{\infty} D^k \frac{t^k}{k!}\right)T^{-1} = Te^{Dt}T^{-1}$$

4. (22 pts) Consider the state dynamics below in which T is a square invertible matrix,

$$\dot{x} = T\left[\begin{array}{c|c} -I & A_{12} \\ \hline 0 & -I \end{array}\right]T^{-1}x + T\left[\begin{array}{c} B_1 \\ \hline 0 \end{array}\right]u$$

(a) (12 pts) Compute e^{At} and the zero-input response to the initial condition $x(t_0) = x_0$.
(b) (10 pts) Compute the zero-state response to the input $u(t) = v \times 1^+(t)$ for some constant vector $v \in R^m$.

SOLUTION 4. (a) (12 pts) $e^{At} = Te^{Dt}T^{-1}$ where $D = \left[\begin{array}{c|c} -I & A_{12} \\ \hline 0 & -I \end{array}\right] = -\left[\begin{array}{c|c} I & 0 \\ \hline 0 & I \end{array}\right] + \left[\begin{array}{c|c} 0 & A_{12} \\ \hline 0 & 0 \end{array}\right]$. The matrices in the decomposition commute. Therefore $e^{Dt} = e^{-t}\left[\begin{array}{c|c} I & 0 \\ \hline 0 & I \end{array}\right]\exp\left(\left[\begin{array}{c|c} 0 & A_{12} \\ \hline 0 & 0 \end{array}\right]t\right) = e^{-t}\left[\begin{array}{c|c} I & A_{12}t \\ \hline 0 & I \end{array}\right]$.

Finally

$x(t) = e^{-(t-t_0)}T\left[\begin{array}{c|c} I & A_{12}(t-t_0) \\ \hline 0 & I \end{array}\right]T^{-1}x_0$.

(b) (10 pts) $e^{A(t-q)}Bu(q) = e^{-(t-q)}T\left[\begin{array}{c|c} I & A_{12}(t-q) \\ \hline 0 & I \end{array}\right]T^{-1}T\begin{bmatrix} B_1 \\ 0 \end{bmatrix}v1^+(q) = e^{-(t-q)}T\begin{bmatrix} B_1 v \\ 0 \end{bmatrix}1^+(q)$. Hence

$$x(t) = \left(\int_0^t e^{-(t-q)}dq\right)T\begin{bmatrix} B_1 v \\ 0 \end{bmatrix} = (1-e^{-t})T\begin{bmatrix} B_1 v \\ 0 \end{bmatrix}$$

5. (12 pts) Develop either the controllable or observable canonical form (your choice) of the differential equation

$$\ddot{y} + a_1\dot{y} + a_2 y = b_2 u + b_1\dot{u} + b_0\ddot{u}$$

See HW solutions for which this problem is quite similar.

6. (10 pts) Consider the state dynamics $\dot{x} = Ax + Bu$ where

$$A = \left[\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ -a_2 & -a_1 & b_1 & b_2 \\ \hline 0 & 0 & 0 & 1 \\ d_1 & d_2 & -c_2 & -c_1 \end{array}\right]; B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Find F in terms of a_i, b_i, c_i, and d_i so that (i) the resulting system is block diagonal, and (ii) the polynomial of the system is $\pi_{A+BF}(\lambda) = (\lambda^2+\lambda+1)(\lambda^2+\lambda+1)$.

SOLUTION:

$$F = \begin{bmatrix} a_2-1 & a_1-1 & -b_1 & -b_2 \\ -d_1 & -d_2 & c_2-1 & c_1-1 \end{bmatrix}$$

EE-602
Exam I
September 17, 2003

130 Point Exam

INSTRUCTIONS

This is a closed book, closed notes exam. You are permitted only a calculator. Work patiently, efficiently, and in an organized manner clearly identifying the steps you have taken to solve each problem. Your grade for each problem depends on the completeness, organization and clarity of your work as well as the resulting answer. All students are expected to abide by the usual ethical standards of the university, i.e., your answers must reflect only your own knowledge and reasoning ability.

There are a total of 12 pages , 6 problems, and one extra credit problem.

Good luck.

Name: ______________________________

1. (20 pts) (a) (3 pts) State the definition of a linear system represented by an operator L:U → Y.
(b) (3 pts) State the definition of a time invariant system represented by an operator N:U → Y.
(c) (9 pts) Prove that the zero-state state-response of the state dynamics, $\dot{x}(t) = A(t)x(t) + B(t)u(t)$, is linear. Clearly state any assumptions and what exactly you must prove.
(d) (5 pts) Suppose a linear time varying causal system has the known response $y_1(t)$ to the input $u(t) = 1^+(t+T) - 1^+(t-T)$ for some T > 0. What can be said of the response $y_2(t)$ to the input $u(t) = K_0 1^+(t+T)$ for some constant K_0.

Solution (c) $u_i \to x_i$ for i = 1, 2, 3 where $u_3 = a_1u_1 + a_2u_2$. For i = 1, 2, 3, $\dot{x}_i(t) = A(t)x_i(t) + B(t)u_i(t)$. We will assume admissible inputs in which case x(-∞) = 0. Now

$$\begin{aligned}\frac{d}{dt}(a_1x_1 + a_2x_2) &= a_1\dot{x}_1 + a_2\dot{x}_2 \\ &= A(t)a_1x_1(t) + B(t)a_1u_1(t) + A(t)a_2x_2(t) + B(t)a_2u_2(t) \\ &= A(t)[a_1x_1(t) + a_2x_2(t)] + B(t)[a_1u_1(t) + a_2u_2(t)]\end{aligned}$$

Hence x_3 and $a_1x_1 + a_2x_2$ satisfy the same DE and IC at $-\infty$. Therefore by the uniqueness theorem they coincide and the linearity is satisfied.

(d). Since the system is linear and causal, for t ≤ -T, the $y_2(t) = 0$ and for –T ≤ t ≤ T, $y_2(t) = K_0 y_1(t)$.

2. (12 pts) A simplified model of the vertical assent of a rocket whose mass changes with time is approximated by

$$\frac{d}{dt}\left[(m_0 + m(t))\frac{dr}{dt}\right] = -(m_0 + m(t))g\left(\frac{R}{r}\right)^2 + T(t)$$

where R in miles is the earth's radius, r is the distance of the rocket above the earth's surface, $[m_0 + m(t)] = m_0 + m_1 e^{-\lambda t}$ is the time varying mass of the rocket with m_0 representing the fixed mass of the rocket, g is the acceleration due to gravity, and T(t) is the thrust, an input. Choose state variables and write state equations for this differential equation. No output equation is needed.

Solution. Expanding the derivative,

$$(m_0 + m(t))\frac{d^2r}{dt^2} + \dot{m}(t)\frac{dr}{dt} = -(m_0 + m(t))g\left(\frac{R}{r}\right)^2 + T(t)$$

Hence,

$$\frac{d^2r}{dt^2} = -\frac{\dot{m}(t)}{(m_0 + m(t))}\frac{dr}{dt} - g\left(\frac{R}{r}\right)^2 + \frac{1}{(m_0 + m(t))}T(t)$$

By inspection with $x_1 = r$ and $x_2 = \dot{x}_1 = \dot{r}$,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -\dfrac{\dot{m}(t)}{(m_0 + m(t))}x_2 - g\left(\dfrac{R}{x_1}\right)^2 + \dfrac{1}{(m_0 + m(t))}T(t) \end{bmatrix}$$

3. (25 pts) The predator-prey model of a biological system with the Bush administration adding quail to the host population in which hunting is open to people with Lugars is:

$$\frac{dH}{dt} = a_1 H - c_1 PH + d_1 u$$
$$\frac{dP}{dt} = -a_2 P + c_2 PH$$

where (i) a_1, a_2, c_1, c_2, and d_1 are all non-zero positive values unequal to each other, (ii) a_1H is the uninhibited host's growth rate, (iii) $-c_1PH$ represents the decrease in growth rate due to the presence of parasites, (iv) $-a_2P$ represents the natural death rate of the parasites, and (v) the product term, c_2HP, models the dependence of the growth rate of the parasites upon the existence of a host.

(a) (10 pts) Determine nonzero constant equilibrium solutions H* (≠ 0) and P* (≠0) given a constant feed rate u* (≠0).

(b) (12 pts) Linearize the system about these nonzero constant equilibrium solutions. Specify the linearized perturbation state model of the system in terms of the variables P* and H*, NOT the values computed in (a).

Solution. **(a)** Consider $0 = -a_2P^* + c_2P^*H^* = P^*(c_2H^* - a_2) \Rightarrow H^* = \frac{a_2}{c_2}$. Hence

$0 = a_1H^* - c_1P^*H^* + d_1u^* \Rightarrow P^* = \frac{a_1}{c_1} + \frac{d_1}{c_1H^*}u^* = \frac{a_1}{c_1} + \frac{d_1c_2}{a_2c_1}u^*$ since H* ≠ 0.

(b) $A = \begin{bmatrix} a_1 - c_1P^* & -c_1H^* \\ c_2P^* & c_2H^* - a_2 \end{bmatrix}$, $B = \begin{bmatrix} d_1 \\ 0 \end{bmatrix}$, $\Delta x = \begin{bmatrix} \Delta H \\ \Delta P \end{bmatrix}$, and $\Delta\dot{x} = A\Delta x + B\Delta u$.

4. (25 pts) Suppose A = T D T^{-1} for an appropriate nonsingular matrix T. (Here D is often diagonal, but can be any matrix.)

(a) (6 pts) Show by induction that $A^k = T D^k T^{-1}$.

(b) (6 pts) Use (a) and the definition of the Taylor series to prove that exp(At) = T exp(Dt) T^{-1}.

(c) (13 pts) Show that for any constant matrix A, $\frac{d}{dt}e^{At} = Ae^{At}$ and then use the uniqueness theorem to prove that if $A_1A_2 = A_2A_1$ then $\exp[A_1t]\exp[A_2t] = \exp[(A_1 + A_2)t]$.

5. (12 pts) (a) (4 pts) State Libnitz rule for differentiation.

(b) (8 pts) Use Libnitz rule to prove that the zero-state state response to $\dot{x}(t) = Ax(t) + Bu(t)$ is

$$\int_{t_0}^{t} e^{A(t-q)} Bu(q) dq$$

See class notes.

6. (40 pts) (a) (6 pts) Consider the block diagonal square A-matrix of the form

$$A \equiv \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline 0 & A_{22} \end{array}\right] = \left[\begin{array}{c|c} A_{11} & 0 \\ \hline 0 & A_{22} \end{array}\right] + \left[\begin{array}{c|c} 0 & A_{12} \\ \hline 0 & 0 \end{array}\right] \equiv M + N$$

where A_{ii} are $n_i \times n_i$ respectively and A_{12} is $n_1 \times n_2$. Under what conditions does exp(At) = exp(Mt)$\times$exp(Nt) where N is nilpotent.
(b) (6 pts) Under the conditions derived in (a), compute an expression for the matrix exponential of A.
(c) Consider the state dynamics

$$\dot{x}(t) = \left[\begin{array}{cc|cc} -\lambda & 0 & 1 & 0 \\ 0 & -\lambda & 1 & 0 \\ \hline 0 & 0 & -\lambda & 0 \\ 0 & 0 & 1 & -\lambda \end{array}\right] x(t) + \left[\begin{array}{cc} 0 & 0 \\ 0 & 0 \\ \hline 1 & 0 \\ 0 & 1 \end{array}\right] u(t)$$

- **(i) (11 pts)** Find e^{At}. Do not multiply out!!!!
- **(ii) (5 pts)** If $x(0) = [0 \quad 0 \quad 1 \quad 0]^T$ find the zero-input response $x_{zi}(t)$. (Multiply out.)
- **(iii) (9 pts)** If $u(t) = [0 \quad 1]^T e^{-\lambda t} 1^+(t)$, find the zero-state response $x_{zs}(t)$.
- **(iv) (3 pts)** What is the complete response?
- **(v) (2 pts)** Compute the natural frequencies of the system.

Solution. (a) MN = NM in which case $A_{11}A_{12} = A_{12}A_{22}$.
(b) By inspection, since $N^2 = [0]$

$$e^{At} = \begin{bmatrix} e^{A_1 t} & 0 \\ 0 & e^{A_2 t} \end{bmatrix} \begin{bmatrix} I & A_{12}t \\ 0 & I \end{bmatrix}$$

(c) (i) $$e^{At} = \left[\begin{array}{cc|cc} e^{-\lambda t} & 0 & 0 & 0 \\ 0 & e^{-\lambda t} & 0 & 0 \\ \hline 0 & 0 & e^{-\lambda t} & 0 \\ 0 & 0 & te^{-\lambda t} & e^{-\lambda t} \end{array}\right] \left[\begin{array}{cc|cc} 1 & 0 & t & 0 \\ 0 & 1 & t & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

(ii) $$x(t) = e^{At}x(0) = \left[\begin{array}{cc|cc} e^{-\lambda t} & 0 & 0 & 0 \\ 0 & e^{-\lambda t} & 0 & 0 \\ \hline 0 & 0 & e^{-\lambda t} & 0 \\ 0 & 0 & te^{-\lambda t} & e^{-\lambda t} \end{array}\right] \left[\begin{array}{cc|cc} 1 & 0 & t & 0 \\ 0 & 1 & t & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} te^{-\lambda t} \\ te^{-\lambda t} \\ e^{-\lambda t} \\ te^{-\lambda t} \end{bmatrix} = \begin{bmatrix} t \\ t \\ 1 \\ t \end{bmatrix} e^{-\lambda t}$$

(iii) $$e^{A(t-q)}Bu(q) = e^{-\lambda(t-q)} \left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & t & 1 \end{array}\right] \left[\begin{array}{cc|cc} 1 & 0 & t & 0 \\ 0 & 1 & t & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} e^{-\lambda q} 1^+(q) = e^{-\lambda t} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} 1^+(q)$$

Hence

$$x(t) = te^{-\lambda t} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$