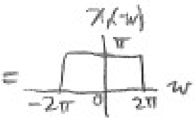
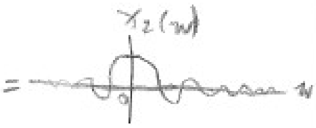
1. Determine whether each of the following signals is band limited. (Answer yes/no and justify.) If they are band limited, specify their Nyquist rate.

(5 pts) a) $x_1(t) = \frac{\sin(2\pi t)}{t} \times \frac{\pi}{\pi} \xrightarrow{\text{By } 19} X_1(w) = \pi\left(u(w+2\pi) - u(w-2\pi)\right) =$ 

yes, $w_m = 2\pi \to$ Nyquist rate $= 2w_m = 4\pi$

(5 pts) b) $x_2(t) = 3\left(u(t+2) - u(t-2)\right) \xrightarrow{\text{By } 21} X_2(w) = 6\,\frac{\sin(w2)}{w} =$ 

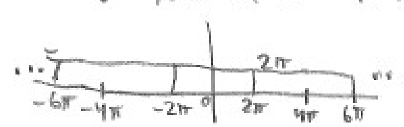
no, no w_m

(5 pts) c) $x_3(t) = \frac{\sin(2\pi t)}{t} \sum_{n=-\infty}^{\infty} \delta(t - \frac{n}{2})$

no, no w_m

By 14,19,22 $\longrightarrow X_3(w) = \frac{1}{2\pi}\left(\pi\left(u(w+2\pi) - u(w-2\pi)\right) * \left(\frac{2\pi}{1/2} \sum_{K=-\infty}^{\infty} \delta\left(w - \frac{2\pi K}{1/2}\right)\right)\right)$

$= 2\pi\left(u(w+2\pi) - u(w-2\pi)\right) * \sum_{K=-\infty}^{\infty} \delta(w - 4\pi K)$

= pulse train 

(5 pts) d) $x_4(t) = \left(\frac{\sin(2\pi t)}{t}\right)^2$

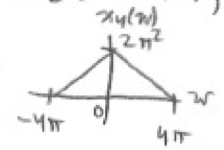
$$\xrightarrow{\text{By } 14, 19} X_4(w) = \frac{1}{2\pi}\left(\pi(u(w+2\pi)-u(w-2\pi)) * \pi(u(w+2\pi) - u(w-2\pi))\right)$$

yes, $w_m = 4\pi$

Nyquist rate $= 2w_m = 8\pi$

$$= \begin{cases} \frac{\pi}{2}(w+4\pi), & -4\pi \le w \le 0 \\ \frac{\pi}{2}(-w+4\pi), & 0 \le w \le 4\pi \end{cases}$$

$=$

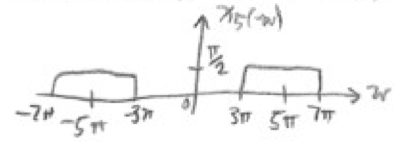


(5 pts) d) $x_5(t) = \frac{\sin(2\pi t)}{t}\cos(5\pi t) = \pi\,\frac{\sin(2\pi t)}{\pi t}\left(\frac{1}{2}e^{j5\pi t} + \frac{1}{2}e^{-j5\pi t}\right)$

$$\xrightarrow{\text{By } 14, 17, 19} X_5(w) = \frac{1}{2\pi}\left(\pi(u(w+2\pi)-u(w-2\pi)) * (\pi\delta(w-5\pi) + \pi\delta(w+5\pi))\right)$$

$$= \frac{\pi}{2}(u(w+2\pi) - u(w-2\pi)) * (\delta(w-5\pi) + \delta(w+5\pi))$$

$=$



$\rightarrow$ yes, $w_m = 7\pi$

Nyquist rate $= 2w_m = 14\pi$

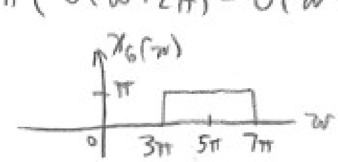
(5 pts) e) $x_6(t) = \frac{\sin(2\pi t)}{t}e^{j5\pi t}$

By 14, 17, 19

$$\rightarrow X_6(w) = \frac{1}{2\pi}\left(\pi(u(w+2\pi) - u(w-2\pi)) * (2\pi\,\delta(w-5\pi))\right)$$

$$= \pi(u(w+2\pi) - u(w-2\pi)) * \delta(w-5\pi)$$

$=$



$\rightarrow$ yes, $w_m = 7\pi \rightarrow$ Nyquist rate $= 2w_m = 14\pi$

(15 pts) **2.** Using the definition of the z-transform (i.e. do not simply take the answer from the table), obtain the z-transform (with its ROC) of

$$x[n] = 3^n u[-n-1]$$

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} = \sum_{n=-\infty}^{\infty} 3^n z^{-n} u[-n-1]$$

$$u[-n-1] = \begin{cases} 1, & -n-1 \geq 0 \\ 0, & -n-1 < 0 \end{cases} = \begin{cases} 1, & n \leq -1 \\ 0, & n > -1 \end{cases}$$

$$\rightarrow X(z) = \sum_{n=-\infty}^{-1} \left(\frac{3}{z}\right)^n \quad \rightarrow \text{Let } m = -n$$

$$\rightarrow X(z) = \sum_{m=1}^{\infty} \left(\frac{z}{3}\right)^m \quad \rightarrow \text{Let } k = m-1 \rightarrow m = k+1$$

$$\rightarrow X(z) = \sum_{k=0}^{\infty} \left(\frac{z}{3}\right)^{k+1} = \frac{z}{3} \sum_{k=0}^{\infty} \left(\frac{z}{3}\right)^k = \frac{z}{3} \cdot \frac{1}{1-\frac{z}{3}} \text{ in the ROC}$$

$$\text{ROC: } -1 < \frac{z}{3} < 1 \rightarrow -3 < z < 3$$

$$\rightarrow X(z) = \begin{cases} \frac{z}{3} \cdot \frac{1}{1-\frac{z}{3}}, & -3 < z < 3 \\ \text{divergent}, & \text{else} \end{cases}$$

(20 pts) **3.** The Laplace transform of the unit impulse response of a system is

$$H(s) = \frac{1}{s+2}, \mathrm{Re}(s) > -2.$$

Determine the response $y(t)$ of the system when the input is

$$x(t) = \begin{cases} e^{-3t} & t > 0 \\ 0 & t = 0 \\ -e^{3t} & t < 0 \end{cases}.$$

not needed

since $\mathrm{Re}(s) > -2$ contains the imaginary axis, the Fourier Transform is in the ROC.

$$\rightarrow \mathcal{H}(\omega) = H(j\omega) = \frac{1}{2+j\omega}$$

$$X(s) \overset{\text{By } S2, S3}{=} \begin{cases} \frac{1}{s+3} & , \mathrm{Re}(s) > -3 \\ \frac{1}{s-3} & , \mathrm{Re}(s) < 3 \\ \text{divergent} & , \text{else} \end{cases} = \begin{cases} \frac{1}{s+3} + \frac{1}{s-3} & , -3 < \mathrm{Re}(s) < 3 \\ \text{divergent} & , \text{else} \end{cases}$$

$$Y(s) = H(s)X(s) = \frac{1}{s+2}\left(\frac{1}{s+3} + \frac{1}{s-3}\right) = \frac{1}{(s+2)(s+3)} + \frac{1}{(s+2)(s-3)}$$

$$= \frac{A}{s+2} + \frac{B}{s+3} + \frac{C}{s+2} + \frac{D}{s-3} = \frac{1}{s+2} - \frac{1}{s+3} - \frac{1}{5}\frac{1}{s+2} + \frac{1}{5}\frac{1}{s-3} = \frac{4}{5}\frac{1}{s+2} - \frac{1}{s+3} + \frac{1}{5}\frac{1}{s-3}$$

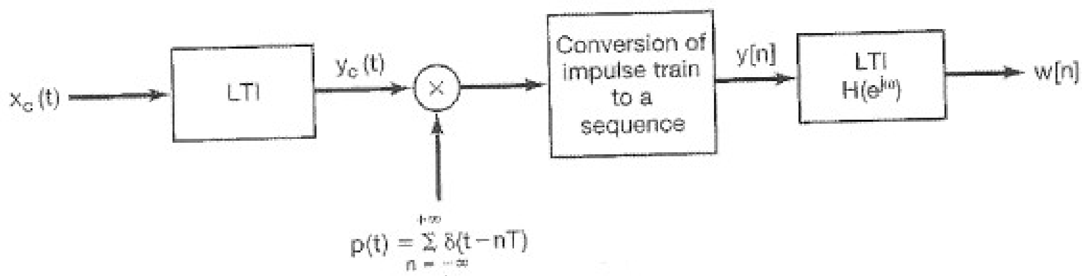
By S3

$$\rightarrow y(t) = \frac{4}{5}e^{-2t}u(t) - e^{-3t}u(t) - \frac{1}{5}e^{3t}u(-t)$$

4. The block diagram below shows a system consisting of a continuous-time LTI system followed by a sampler, conversion to a sequence, and an LTI discrete-time system. The continuous-time LTI system is causal and satisfies the linear, constant-coefficient differential equation

$$\frac{dy_c(t)}{dt} + y_c(t) = x_c(t).$$

The input $x_c(t)$ is a unit impulse $\delta(t)$.



(10 pts) **a)** Determine the input $y_c(t)$.

$X_c(s) \overset{\text{By 54}}{=} 1$

$\mathcal{L}\left(\frac{d}{dt}y_c(t) + y_c(t)\right) = \mathcal{L}(x_c(t)) \overset{\text{By 46}}{\rightarrow} \quad sY_c(s) + Y_c(s) = 1$

$\rightarrow Y_c(s) = \frac{1}{s+1} \quad \rightarrow \quad y_c(t) = \mathcal{L}^{-1}\left(\frac{1}{s+1}\right) = e^{-t}u(t)$ or $-e^{-t}u(-t)$ depending on ROC

since causal: impulse response is 0 for $t<0$

$\rightarrow y_c(t) = e^{-t}u(t)$

(Problem 7 continues on the next page.)

(15 pts) **b)** Determine the frequency response $H(e^{j\omega})$ and the unit impulse response $h[n]$ such that $w[n] = \delta[n]$.

$y[n] = y(nT) = e^{-nT} u(nT) = e^{-nT} u[n] = (e^{-T})^n u[n]$

$\rightarrow Y(w) \overset{\text{By 38}}{=} \dfrac{1}{1 - e^{-T} \cdot e^{-jw}} = \dfrac{1}{1 - e^{-T-jw}}$, for $e^{-T} < 1 \rightarrow T > 0$ (which it is)

$W(w) \overset{\text{By 36}}{=} 1$

$H(w) = \dfrac{W(w)}{Y(w)} = 1 - e^{-T-jw} = 1 - e^{-T} e^{-jw}$

By 26, 36 $\rightarrow$ $h[n] = \delta[n] - e^{-T} \delta[n-1]$