3.6 DT Fourier Series (Periodic Signals)

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3:07 PM

z^n

$$(\sqrt{2}j)^n \rightarrow \boxed{LTI} \rightarrow H(\sqrt{2}j)\cdot(\sqrt{2}j)^n$$

$$\sum a_k e^{jk\omega_0 n} \rightarrow \boxed{LTI} \rightarrow \sum a_k H(e^{jk\omega_0})\, e^{jk\omega_0 n}$$

Use Fourier to write signals as sumations of complex exponentials

Fourier Series Formula

Let $x[n]$ be a periodic DT Signal with fundamental Period N

Write $\omega_0 = \dfrac{2\pi}{N}$

Then $$x[n] = \sum_{\boxed{k=0}}^{\boxed{N-1}} a_k e^{jk\omega_0 n}, \quad \text{where } a_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n]\, e^{-jk\omega_0 n}$$

why a finite number in sum?

Because the set $\{e^{jk\omega_0 n}\}_{k\in\mathbb{Z}}$ (harmonically related exponentials)

contains a finite number of distinct functions.

repeats

$k=0$ $k=1$ $k=2$ $k=N$ $k=N+1$

$e^0=1$ $e^{j\omega_0 n}$ $e^{2j\omega_0 n}$ $e^{2Nj\omega_0 n} = e^{2Nj\omega_0 \frac{2\pi}{N} n} =$ $e^{(N+1)j\omega_0 n} \Rightarrow e^{j\omega_0 n}$

$= e^{j2\pi n} = 1$

it repeats so you only need to go from $0 \Rightarrow N-1$ to get all values.

Question: Obtain the DT Fourier series coefficients (a_k) of the signal

$x[n] = \sin(3\pi n + \frac{\pi}{2})$

$N = \frac{2\pi}{3\pi} \cdot k = \frac{2}{3} k$, take $k=3$ then $N=2$ *important to know

$x[0] = 1$
$x[1] = -1$
$x[2] = 1$
$x[3] = -1$

Note: the function $x[n] = (-1)^n$ is same $\sin(3\pi n + \frac{\pi}{2})$ in DT

$$a_k = \frac{1}{2} \sum_{n=0}^{1} x[n] e^{-jk\pi n}$$

$$a_0 = \frac{1}{2} \sum_{n=0}^{1} x[n] e^{0} = 0$$

$$a_1 = \frac{1}{2} 1 e^0 + \frac{1}{2}(-1) e^{-j\pi} = \frac{1}{2} + \frac{-1}{2} e^{-j\pi} = \frac{1}{2} - \frac{1}{2}(-1) = 1$$

$x[n] = 0 \cdot e^0 + 1 e^{j\pi n} = e^{j\pi n}$ which is same as $\sin(3\pi + \frac{\pi}{2})$

Another way to get a_k's

$$x[n] = \sin(3\pi n + \tfrac{\pi}{2}) \overset{\text{euler's}}{=} \frac{e^{j(3\pi n + \frac{\pi}{2})} - e^{-j(3\pi n + \frac{\pi}{2})}}{2j}$$

split

$$= \frac{1}{2j} \underbrace{e^{j\frac{\pi}{2}}}_{j} e^{j3\pi n} - \frac{1}{2j} \underbrace{e^{-j\frac{\pi}{2}}}_{-j} e^{-j3\pi n}$$

$$= \frac{1}{2} e^{j3\pi n} + \frac{1}{2} e^{-j3\pi n}$$

add -2π (one period) add 4π (2 periods)

because $e^{j\pi}$ is periodic with period 2π

$$= \frac{1}{2} e^{j\pi n} + \frac{1}{2} e^{j\pi n}$$

$$= 1 e^{j\pi n}$$

$k=1$ so $a_{k=1}$

No $k=0$ term so $a_k = 0$