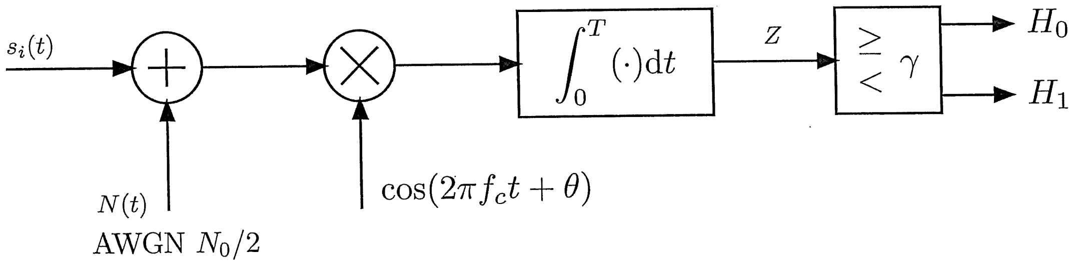
ECE 544 Exam 1 10/14/13

Name: Solution

General Instructions:

- **You have 90 minutes to complete the exam.**
- Write your name on every page of the exam.
- Do not write on the backs of the pages. If you need more paper, it will be provided to you upon request.
- The exam is closed book and closed notes.
- Calculators are allowed.
- Your work must be explained to receive full credit.
- Point values for each problem are as indicated. The exam totals 100 points.
- All plots must be carefully drawn with axes labeled.

Do not open the exam until you are told to begin.



Problem 1. [40 pts. total] Consider the correlation receiver shown above. Assume that the two signals $s_0(t)$ and $s_1(t)$ are equally likely with $s_1(t) = 0$ and

$$s_0(t) = Ap_T(t)\cos(2\pi f_c t + \theta),$$

where $p_T(t) = 1$, for $0 \leq t \leq T$, and $p_T(t) = 0$, for $t < 0$ or $t > T$. Also assume that $f_c \gg T^{-1}$.

In this problem we will step through the choosing of an optimal threshold and the evaluation of the performance of the above communication system. For full credit you must work the steps as instructed.

(a) [10 pts.] Under the assumption that s_0 was transmitted (i.e., H_0) find the form of the probability distribution of the random variable Z and specify any parameters of the distribution in terms of the constants given in the problem statement. Explain.

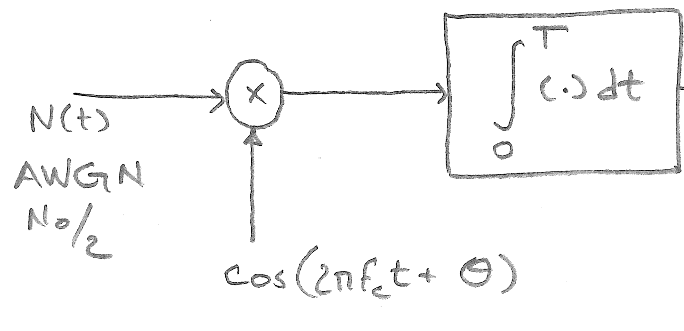
(b) [10 pts.] Repeat part (a) assuming that s_1 was transmitted (i.e., H_1).

(c) [10 pts.] For a generic threshold γ find expressions for the conditional error probabilities $P_{e,0}$ and $P_{e,1}$ and write them in terms of the Gaussian Q function, the problem constants, and γ. Explain.

(d) [10 pts.] Find the value of the threshold γ (i.e., derive it), which minimizes the average probability of error. Also find the resulting minimum average probability of error. Explain.

Problem 1

Let N denote the correlator output when no signal is present:



$$N = \int_0^T N(t)\cos(2\pi f_c t + \theta)\,dt$$

Easy to see that N is a Gaussian rv of mean zero. For its variance

$$E\{N^2\} = E\left\{ \int_0^T N(t)\cos(2\pi f_c t+\theta)\,dt \int_0^T N(s)\cos(2\pi f_c s+\theta)\,ds \right\}$$

$$= \int_0^T\int_0^T E\{N(t)N(s)\}\cos(2\pi f_c t+\theta)\cos(2\pi f_c s+\theta)\,dt\,ds$$

$$= \int_0^T\int_0^T \frac{N_0}{2}\delta(s-t)\cos(2\pi f_c t+\theta)\cos(2\pi f_c s+\theta)\,dt\,ds$$

$$= \frac{N_0}{2}\int_0^T \cos^2(2\pi f_c t+\theta)\,dt$$

$$= \frac{N_0}{2}\int_0^T \left[\frac{1+\cos(2\pi\, 2f_c t + 2\theta)}{2}\right]dt$$

$$\simeq \frac{N_0}{4}T \qquad \therefore\ N \sim \mathcal{N}\left(0, \frac{N_0 T}{4}\right)$$

(a) Under H_0 the signal part of Z is

$$A\int_0^T \cos^2(2\pi f_c t + \Theta)\,dt \cong \frac{AT}{2}$$ following the prev. result.

$$\therefore \; Z \sim N\left(\frac{AT}{2}, \frac{N_0 T}{4}\right)$$

(b) Under H_1 the signal part is zero.

Therefore

$$Z \sim N\left(0, \frac{N_0 T}{4}\right)$$

(c) The test is

$$Z \begin{array}{c} \geq \\ < \end{array} \gamma \quad \begin{array}{l} \text{say } H_0 \\ \text{say } H_1 \end{array}$$

$$P_{e,0} = P_0(Z < \gamma)$$

$$= P\left(\frac{AT}{2} + N < \gamma\right) = P\left(N < \gamma - AT/2\right)$$

$$= P\left(\frac{N}{\sqrt{N_0 T/4}} < \frac{\gamma - AT/2}{\sqrt{N_0 T/4}}\right) = \Phi\left(\frac{\gamma - AT/2}{\sqrt{N_0 T/4}}\right)$$

$$= Q\left(\frac{AT/2 - \gamma}{\sqrt{N_0 T/4}}\right)$$

$$P_{e,1} = P_1(Z \geq \gamma) = P(N \geq \gamma)$$

$$= P\left(\frac{N}{\sqrt{N_0 T/4}} \geq \frac{\gamma}{\sqrt{N_0 T/4}}\right)$$

$$= Q\left(\frac{\gamma}{\sqrt{N_0 T/4}}\right)$$

(d) For equal priors the solution is minimax so must solve for γ st.

$$Q\left(\frac{\gamma}{\sqrt{N_0 T/4}}\right) = Q\left(\frac{AT/2 - \gamma}{\sqrt{N_0 T/4}}\right)$$

which gives

$$\gamma_{opt} = AT/4$$

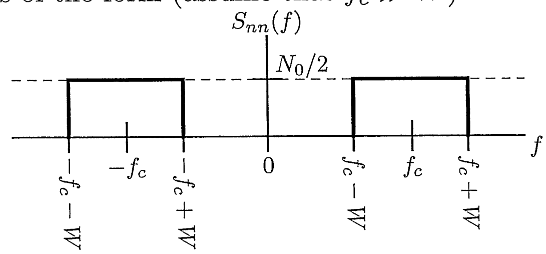
Then

$$P_e = Q\left(\frac{AT/4}{\sqrt{N_0 T/4}}\right) = Q\left(\sqrt{\frac{A^2T^2}{16} \cdot \frac{4}{N_0 T}}\right)$$

$$= Q\left(\sqrt{\frac{A^2 T}{4 N_0}}\right)$$

Solution

Problem 2. [40 pts. total] Let $n(t)$ be a wide-sense stationary (WSS), zero-mean random process obtained by filtering white Gaussian noise with an ideal bandpass filter. That is, its power spectral density is of the form (assume that $f_c \gg W$):



The purpose of this problem is to work out a small part of the proof of the theorem below:

Theorem: Let $n(t)$ be as above. Then it may be expressed in the form

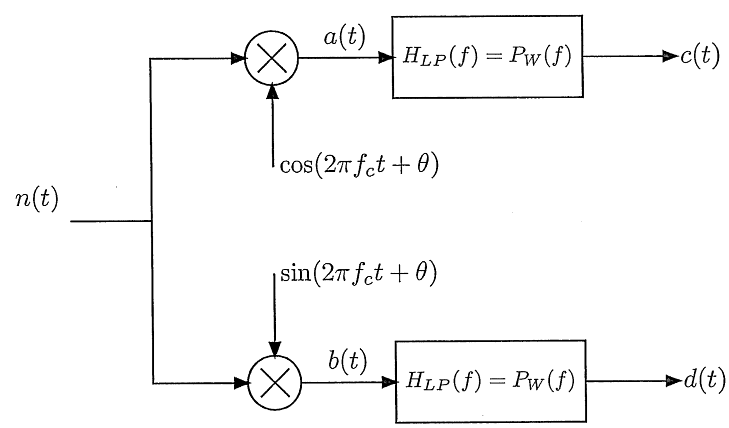
$$n(t) = x(t)\cos(2\pi f_c t + \theta) + y(t)\sin(2\pi f_c t + \theta)$$

where: 1) the equality is interpreted in mean-square[1], 2) $x(\cdot)$ and $y(\cdot)$ are jointly WSS, Gaussian, zero-mean with identical psds, which do not depend on θ, and 3) $x(\cdot)$ and $y(\cdot)$ are statistically independent random processes.

For notational simplicity define $P_W(f) = 1$, for $|f| \le W$, and $P_W(f) = 0$, for $|f| > W$, let $p_W(t) \leftrightarrow P_W(f)$, and write

$$S_{nn}(f) = \frac{N_0}{2} P_W(f - f_c) + \frac{N_0}{2} P_W(f + f_c).$$

For the questions on the pages to follow refer to the down-converter shown below.



[1] i.e.,

$$\mathrm{E}\{[n(t) - x(t)\cos(2\pi f_c t + \theta) - y(t)\sin(2\pi f_c t + \theta)]^2\} = 0$$

for all t.

Problem 2. (cont'd.)

(a) [5 pts.] With $R_{ab}(t,s) = \mathrm{E}\{a(t)b(s)\}$ and similarly for $R_{cd}(t,s)$ show that

$$R_{cd}(t,s) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{ab}(u,v)p_W(t-u)p_W(s-v)\mathrm{d}u\mathrm{d}v.$$

$$c(t) = a * p_W(t) = \int_{-\infty}^{\infty} a(u)\, p_W(t-u)\, du$$

$$d(s) = b * p_W(s) = \int_{-\infty}^{\infty} b(v)\, p_W(s-v)\, dv$$

$$R_{cd}(t,s) = E\{c(t)\, d(s)\} = E\left\{ \int_{-\infty}^{\infty} a(u)\, p_W(t-u)\, du \cdot \int_{-\infty}^{\infty} b(v)\, p_W(s-v)\, dv \right\}$$

$$= E\left\{ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} a(u)\, b(v)\, p_W(t-u)\, p_W(s-v)\, du\, dv \right\}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{ab}(u,v)\, p_W(t-u)\, p_W(s-v)\, du\, dv$$

where we're assumed that changing order of integrations are allowed (note: expectation is also an integration) as in Fubini's theorem.

Problem 2. (cont'd.)

(b) [5 pts.] By writing the double integral from part (a) as an iterated integral show that it can be viewed as the cascade of two LTI systems, one operating on the "s" variable and the other operating on the "t" variable as in the block diagram:

$$R_{ab}(t,s) \longrightarrow \boxed{p_W(s)} \xrightarrow{R_{ad}(t,s)} \boxed{p_W(t)} \longrightarrow R_{cd}(t,s)$$

$$R_{cd}(t,s) = \int_{-\infty}^{\infty} \underbrace{\left[\int_{-\infty}^{\infty} R_{ab}(u,v)\, p_W(s-v)\, dv \right]}_{} p_W(t-u)\, du$$

$$\left[R_{ab}(u,\bullet) * p_W \right](s)$$

ie convolution in the second variable evaluated at s and parameterized by u.

$$\triangleq R_{ad}(u,s)$$

$$\Longrightarrow R_{cd}(t,s) = \int_{-\infty}^{\infty} R_{ad}(u,s)\, p_W(t-u)\, du$$

$$= \left[R_{ad}(\bullet,s) * p_W \right](t)$$

ie convolution in the 1st variable evaluated at t and parameterized by s.

This is the meaning of the above cascade of LTI systems.

Problem 2. (cont'd.)

(c) [10 pts.] Find $R_{ab}(t,s)$ in terms of the sinusoids and the autocorrelation $R_{nn}(t,s) = R_{nn}(s-t)$.

$$R_{ab}(t,s) = E\{a(t)\,b(s)\}$$

$$= E\{n(t)\cos(2\pi f_c t + \theta)\, n(s)\sin(2\pi f_c s + \theta)\}$$

$$= \underbrace{E\{n(t)\,n(s)\}}_{R_{nn}(t,s) = R_{nn}(s-t)} \underbrace{\cos(2\pi f_c t + \theta)\sin(2\pi f_c s + \theta)}_{\frac{1}{2}\sin[2\pi f_c(s-t)] + \frac{1}{2}\sin[2\pi f_c(s+t) + 2\theta]}$$

$$\Longrightarrow$$

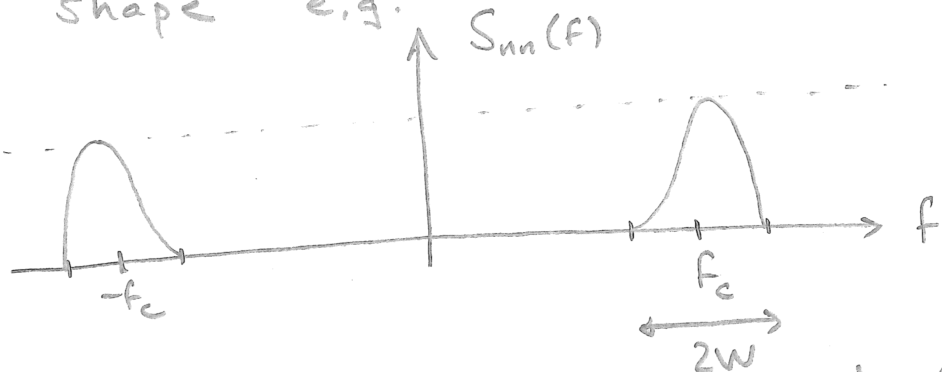
$$R_{ab}(t,s) = \frac{1}{2}R_{nn}(s-t)\sin[2\pi f_c(s-t)] + \frac{1}{2}R_{nn}(s-t)\sin[2\pi f_c(s+t) + 2\theta]$$

A Modest Generalization of the Problem

Problem originally defines

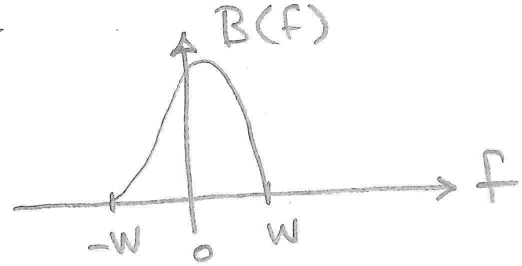
$$S_{nn}(f) = \frac{N_0}{2} P_W(f-f_c) + \frac{N_0}{2} P_W(f+f_c)$$

but more generally suppose that $S_{nn}(f)$ is only non-zero in the vicinity of f_c without imposing the above rectangular shape e.g.



Note that $S_{nn}(f) \geq 0$ since its a psd and $S_{nn}(f) = S_{nn}(-f)$ since the auto correlation is real-valued.

Define $B(f) = P_W(f)\, S_{nn}(f+f_c)$. For above picture it will look like



In general then $B(f) \neq B(-f)$ though it is real-valued.

If $b(t) \longleftrightarrow B(f)$ then $b(t)$ is complex-valued.

Now $P_W(f)\, S_{nn}(f-f_c) = P_W(-f)\, S_{nn}(-f+f_c) = B(-f)$

Problem 2. (cont'd.)

(d) [20 pts.] Use Fourier Transform methods to find $R_{cd}(t,s)$ by carrying out the following steps.

1. Find the Fourier Transform of $R_{ab}(t,s)$ with respect to s and let ν denote the frequency variable. Multiply the resulting transform by $P_W(\nu)$.
2. Find the inverse Fourier Transform of the product above in order to find $R_{ad}(t,s)$.
3. Find the Fourier Transform of $R_{ad}(t,s)$ with respect to t and let f denote the frequency variable. Multiply the resulting transform by $P_W(f)$.
4. Find the inverse Fourier Transform of the product above in order to find $R_{cd}(t,s)$.

(1)

$$R_{nn}(s) \xleftrightarrow{s,\nu} S_{nn}(\nu)$$

$$R_{nn}(s-t) \xleftrightarrow{s,\nu} e^{-j2\pi\nu t} S_{nn}(\nu)$$

$$\sin[2\pi f_c(s-t)] = \frac{e^{j2\pi f_c s}e^{-j2\pi f_c t} - e^{-j2\pi f_c s}e^{+j2\pi f_c t}}{j2}$$

Then using the modulation property of the (s,ν)-Fourier transform:

$$\tfrac{1}{2}R_{nn}(s-t)\sin[2\pi f_c(s-t)]$$

$$\updownarrow s,\nu$$

$$\frac{e^{-j2\pi(\nu-f_c)t}S_{nn}(\nu-f_c)\,e^{-j2\pi f_c t} - e^{-j2\pi(\nu+f_c)t}S_{nn}(\nu+f_c)\,e^{+j2\pi f_c t}}{j4}$$

$$\sin[2\pi f_c(s+t)+2\theta] = \frac{e^{j2\pi f_c s}e^{j2\pi f_c t}e^{j2\theta} - e^{-j2\pi f_c s}e^{-j2\pi f_c t}e^{-j2\theta}}{j2}$$

In the same fashion:

$$\tfrac{1}{2}R_{nn}(s-t)\sin[2\pi f_c(s+t)+2\theta] \xleftrightarrow{s,\nu} \frac{e^{-j2\pi(\nu-f_c)t}S_{nn}(\nu-f_c)\,e^{j2\pi f_c t}e^{j2\theta} - e^{-j2\pi(\nu+f_c)t}S_{nn}(\nu+f_c)\,e^{-j2\pi f_c t}e^{-j2\theta}}{j4}$$

Then the FT-(s,ν) of $R_{ab}(t,s)$ is the sum of the two terms above. We multiply by $P_W(\nu)$ and use

$$P_W(\nu)S_{nn}(\nu+f_c) = B(\nu); \quad P_W(\nu)S_{nn}(\nu-f_c) = B(-\nu)$$

Problem 2. (cont'd.)

Then $P_W(\nu)\, \mathcal{F}_{s,\nu}\left\{R_{ab}(t,s)\right\}(\nu) =$

$$\frac{e^{-j2\pi(\nu-f_c)t} B(-\nu)\, e^{-j2\pi f_c t} - e^{-j2\pi(\nu+f_c)t} B(\nu)\, e^{+j2\pi f_c t}}{j4}$$

$$+ \quad \frac{e^{-j2\pi(\nu-f_c)t} B(-\nu)\, e^{j2\pi f_c t}\, e^{j2\theta} - e^{-j2\pi(\nu+f_c)t} B(\nu)\, e^{-j2\pi f_c t}\, e^{-j2\theta}}{j4}$$

$$= \quad \frac{e^{-j2\pi\nu t}\left[B(-\nu) - B(\nu)\right]}{j4}$$

$$+ \; e^{-j2\pi\nu t}\left[\frac{B(-\nu)\, e^{j(2\pi 2f_c t + 2\theta)} - B(\nu)\, e^{-j(2\pi 2 f_c t + 2\theta)}}{j4}\right]$$

Want to take the inverse FT wrt s,ν but 1st should simplify a little. Note that $B(\nu)$ is real-valued.

Define even and odd parts

$$B_e(\nu) = \frac{B(\nu) + B(-\nu)}{2} \qquad B_o(\nu) = \frac{B(\nu) - B(-\nu)}{2}$$

Also let

$$\phi \triangleq 2\pi 2 f_c t + 2\theta$$

for shorthand.

Then $P_W(\nu)\, \mathcal{F}_{s,\nu}\{R_{ab}(t,s)\}(\nu) =$

$$-\frac{B_o(\nu)}{j2} e^{-j2\pi\nu t} + \frac{e^{-j2\pi\nu t}}{j4}\Big[B(-\nu)\cos\phi + jB(-\nu)\sin\phi$$
$$-B(\nu)\cos\phi + jB(\nu)\sin\phi\Big]$$

$$= -\frac{B_o(\nu)}{j2} e^{-j2\pi\nu t} - \frac{e^{-j2\pi\nu t}}{j2}\Big[B_o(\nu)\cos\phi - jB_e(\nu)\sin\phi\Big]$$

Can show (using fact that $B(\nu)$ is real-valued)

$$b_I(s) \triangleq \mathrm{Re}\, b(s) \overset{s,\nu}{\longleftrightarrow} B_e(\nu)$$
$$jb_Q(s) \triangleq j\,\mathrm{Im}\, b(s) \overset{s,\nu}{\longleftrightarrow} B_o(\nu)$$

(2) Then take the inverse FT (s,ν) to conclude

$$R_{ad}(t,s) = -j\frac{b_Q(s-t)}{j2} - j\frac{b_Q(s-t)}{j2}\cos\phi$$
$$+ \frac{b_I}{2}(s-t)\sin\phi$$

$$= -\tfrac{1}{2} b_Q(s-t) - \tfrac{1}{2} b_Q(s-t)\cos\left[2\pi 2f_c t + 2\theta\right]$$
$$+ \tfrac{1}{2} b_I(s-t)\sin\left[2\pi 2f_c t + 2\theta\right]$$

(3) Next we take (t,f) – Fourier transform of above and multiply by $P_W(f)$. Since $b_Q(\cdot)$ and $b_I(\cdot)$ are Lowpass the second and third terms corresp. to the above will be zeroed out by application of the LPF $P_W(f)$.

Problem 2. (cont'd.)

The only term that can possibly remain would be due to the first term in $R_{ad}(t,s)$ ... hence we concentrate on that

$$\mathcal{F}_{t,f}\{b_Q(s-t)\}(f) = \int_{-\infty}^{\infty} b_Q(s-t)\, e^{-j2\pi f t}\, dt$$

Let $u = s-t$, $du = -dt$

$$\Longrightarrow \quad = \int_{-\infty}^{\infty} b_Q(u)\, e^{-j2\pi f(s-u)}\, du = e^{-j2\pi f s}\int_{-\infty}^{\infty} b_Q(u)\, e^{j2\pi f u}\, du$$

$$= e^{-j2\pi f s}\left[-j B_o(-f)\right] = j e^{-j2\pi f s} B_o(f)$$

Since $B_o(f)$ is low pass: $j e^{-j2\pi f s} B_o(f) P_W(f) = j e^{-j2\pi f s} B_o(f)$

$$\Longrightarrow P_W(f)\, \mathcal{F}_{t,f}\{R_{ad}(t,s)\}(f) = -\frac{j}{2} e^{-j2\pi f s} B_o(f)$$

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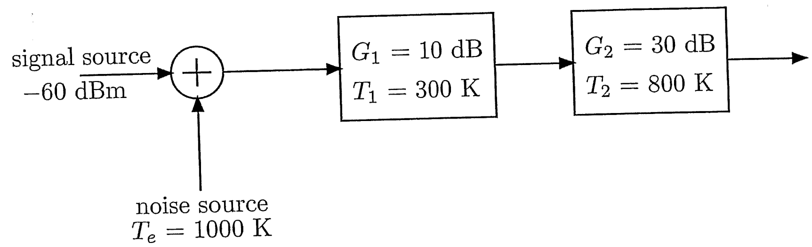
Then taking inverse (t,f) Fourier transform

$$R_{cd}(t,s) = -\frac{j}{2}\, j b_Q(t-s) = \frac{1}{2} b_Q(t-s)$$

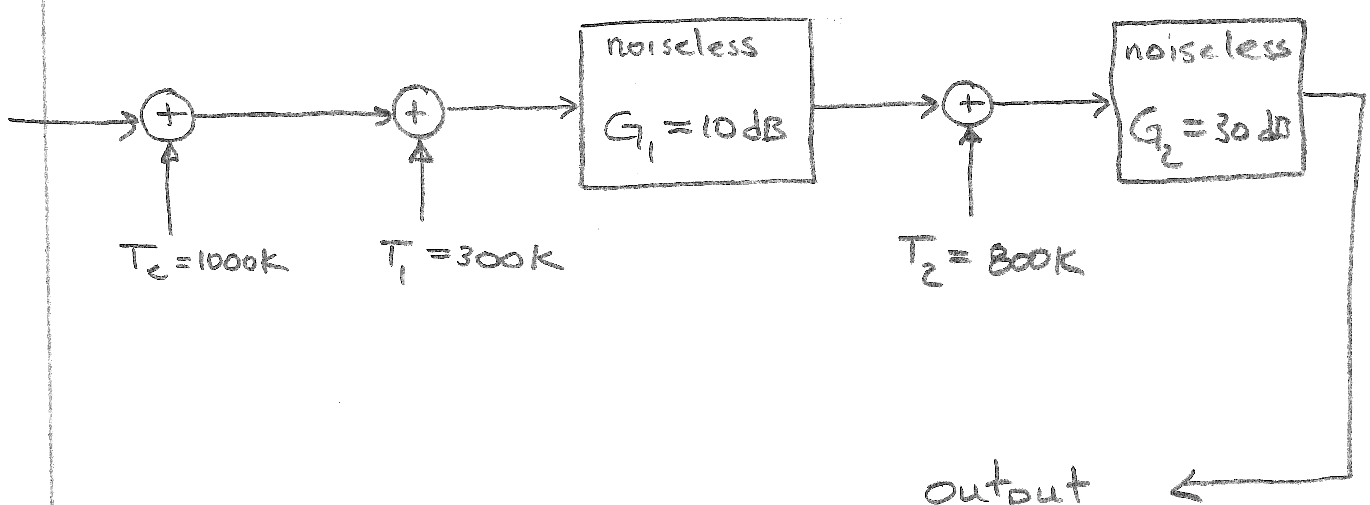
Note: For the actual problem we have $B(f) = \frac{N_o}{2} P_W(f)$ for which $B_o(f) = 0$ and therefore $b_Q(t) = 0$

$\Longrightarrow R_{ed}(t,s) = 0 \quad \forall t,s$



Problem 3. [20 pts. total] Assuming a bandwidth of 100 kHz in the system above find the signal-to-noise power ratio at the output in dB. Boltzmann's constant is $k = 1.38 \times 10^{-23}$ J/K.

Recalling that amplifier noise equiv. temperature refers to an equiv. noise source at the amplifier input can redraw system as:



Since a noise source corresponds to a Gaussian white rp (indep. of all other noise sources) of psd height (2-sided)

$$\frac{kT_e}{2}$$

Then in a bandwidth (1-sided) of B Hz the total output noise power would be

$$\frac{k(T_e + T_1)}{\cancel{2}} \cancel{2}B \cdot G_1 \cdot G_2 + \frac{kT_2}{\cancel{2}} \cancel{2}B \cdot G_2$$

$$N = kB\left[(T_e + T_1)G_1 + T_2\right] G_2$$

where $\quad k = 1.38 \times 10^{-23}$ J/K

$$G_{1\,dB} = 10 \log_{10} G_1 = 10 \text{ dB}$$

$$G_{2\,dB} = 10 \log_{10} G_2 = 30 \text{ dB}$$

$$\Longrightarrow \quad G_1 = 10^{10/10} = 10$$

$$G_2 = 10^{30/10} = 1000$$

$B = 10^5$ Hz

$$\therefore \; N = (1.38 \times 10^{-23})(10^5)\left[1300 \cdot 10 + 800\right] 1000$$

$$= 1.904 \times 10^{-11} \text{ W}$$

$$= -107.2 \text{ dBW}$$

$$= -77.2 \text{ dBm}$$

Signal power at the output is

$$-60 + 10 + 30 = -20 \text{ dBm}$$

$$\therefore \; SNR_{dB} = -20 - (-77.2)$$

$$= 57.2 \text{ dB}$$