EE 301 HW2

1. O+W 1.3 (a, b, d, f): Find P_∞ and E_∞

a) $x_1(t) = e^{-2t} u(t)$

$$E_\infty = \int_0^\infty |x_1(t)|^2 \, dt = \int_0^\infty e^{-4t} \, dt = -\frac{1}{4} e^{-4t} \Big|_{t=0}^{\infty}$$

$$= \frac{1}{4} < \infty \implies P_\infty = 0$$

b) $x_2(t) = e^{j(2t+\pi/4)}$

$$|x_2(t)| = 1 \quad \forall t \longrightarrow E_\infty = \infty$$

$$P_\infty = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x_2(t)|^2 \, dt = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} 1 \cdot dt$$

$$= \lim_{T\to\infty} \frac{1}{2T} 2T = 1$$

d) $x_1[n] = \left(\frac{1}{2}\right)^n u[n]$

$$E_\infty = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{2n} = \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n = \frac{1}{1 - 1/4} = \frac{4}{3} < \infty$$

$$\implies P_\infty = 0$$

f) $x_3[n] = \cos\left(\frac{\pi}{4} n\right)$

$$\cos^2\left(\frac{\pi}{4} n\right) = \frac{1 + \cos\left(\frac{\pi}{2} n\right)}{2}$$ $\implies$ has a DC term.

$\therefore E_\infty = \infty$

$$P_\infty = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \frac{1 + \cos\left(\frac{\pi}{2} n\right)}{2}$$

$$= \frac{1}{2}$$ (exactly as shown in EE 201).

2. O+W 1.27 (d, e, f) Determine if various systems are 1) memoryless 2) time invariant 3) linear 4) causal 5) stable

$$x \longrightarrow \boxed{\text{system}} \longrightarrow y$$

d) $$y(t) = \begin{cases} 0 & t < 0 \\ x(t) + x(t-2) & t > 0 \end{cases}$$

This system

- has memory (not memoryless)
- is not time invariant (note $t = 0$ is a "special time" at which syst. changes)
- is linear
- is causal
- is BIBO stable.

e) $$y(t) = \begin{cases} 0 & \text{if } x(t) < 0 \\ x(t) + x(t-2) & \text{if } x(t) \geq 0 \end{cases}$$

This system

- has memory
- is time-invariant
- is not linear (If $x(t) \mapsto y(t)$ note that $-x(t)$ won't necc. map to $-y(t)$).
- is causal
- is BIBO stable.

f) $y(t) = x(t/3)$

This system

- has memory
- is not time invariant
- is linear
- is not causal $\left(y(t = -1\text{ sec}) = x\left(t = -\tfrac{1}{3}\text{ sec}\right)\right)$
- is BIBO stable.

3. O+W 1.28 (a, f, g) Ditto previous problem

$$x \longrightarrow \boxed{\text{system}} \longrightarrow y$$

a) $y[n] = x[-n]$

This system

- has memory
- is not time-invariant
- is linear
- is not causal
- is BIBO stable.

f) $y[n] = \begin{cases} x[n] & n \geq 1 \\ 0 & n = 0 \\ x[n] & n \leq -1 \end{cases}$

- is memoryless
- is not time invariant.
- is linear
- is causal
- is BIBO stable.

g) $y[n] = x[4n+1]$

- has memory
- is not time-invariant
- is linear
- is not causal
- is BIBO stable.

4. <u>O+W 1.35</u>

Show fundamental period of $x[n] = e^{jm(\frac{2\pi}{N})n}$ has fund. period

$$N_0 = N / \gcd(m,N).$$

First note that $x[n] = x[n+N]$ for all n because

$$e^{jm(\frac{2\pi}{N})n} = e^{jm(\frac{2\pi}{N})(n+N)} = e^{jm(\frac{2\pi}{N})n} \underbrace{e^{jm2\pi}}_{1}$$

Thus N is a period for x (maybe not fundamental). If $N_0 \triangleq$ the fundamental period we know that N must be an integer multiple of N_0 (ie N_0 is a divisor of N). So there exists an integer K st

$$N_0 K = N$$

Since N_0 is a period of $x[n]$ we must have $x[n] = x[n+N_0]$ for all n, ie

$$e^{jm(\frac{2\pi}{N})N_0} = 1$$

which happens if and only if

$$m\left(\frac{2\pi}{N}\right)N_0 = 2\pi k$$

for some integer k. Cancelling the 2π factor

$$m\left(\frac{N_0}{N}\right) = k = \text{an integer}$$

Therefore since $N_0 K = N$ $\left(\frac{N_0}{N} = \frac{1}{K}\right)$

$$\frac{m}{K} = k = \text{an integer}$$

$\Longrightarrow$ K is a divisor of m

$\Longrightarrow$ K is a common divisor of m and N.

The biggest possible K gives the fund. (or smallest) period N_0. So use $K = \gcd(m,N) \Longrightarrow N_0 = N/\gcd(m,N)$.

5. O+W 1.36

$$x(t) = e^{j\omega_0 t} \qquad T_0 = \frac{2\pi}{\omega_0}$$

$$x[n] = x(t)\Big|_{t=nT} = e^{j\omega_0 nT} = e^{j2\pi nT/T_0}$$

a) when is $x[n]$ periodic?

Is periodic if and only if there is an integer N st

$$e^{j2\pi NT/T_0} = 1$$

which is if and only if there is also an integer k st

$$\frac{\cancel{2\pi} NT}{T_0} = \cancel{2\pi} k \iff \frac{T}{T_0} = \frac{k}{N}$$

ie $\frac{T}{T_0}$ is a rational number.

b) Suppose $\frac{T}{T_0} = \frac{p}{q}$ (p, q integers)

Then

$$x[n] = e^{j2\pi n p/q} = e^{jp(2\pi/q)n}$$

as in O+W 1.35 (with $N = q$, $m = p$). Thus fund. period

$$= \frac{q}{\gcd(p,q)}$$

According to O+W definition (p28) the fundamental freq is

$$\Omega_0 = 2\pi/\text{fund. period}$$

$$= \frac{2\pi}{q}\gcd(p,q)$$

Now $\frac{T}{T_0} = \frac{p}{q} = T\frac{\omega_0}{2\pi} \implies \frac{2\pi}{q} = \frac{\omega_0 T}{p}$

Ω_0 = fund. freq. of DT complex sinusoid

$$= \frac{\gcd(p,q)}{p}\,\omega_0 T$$

c) Same assumptions as in (b). How many periods of $x(t)$ needed to make one period of $x[n]$?

Takes

$$\frac{q}{\gcd(p,q)} \quad \text{samples}$$

corresponding to

$$\frac{q}{\gcd(p,q)}\,T \quad \text{seconds}$$

Divide by T_0 to get number of $x(t)$ periods

$$= \frac{q}{\gcd(p,q)}\,\frac{T}{T_0} = \frac{p}{\gcd(p,q)}$$

6. O+W 1.42

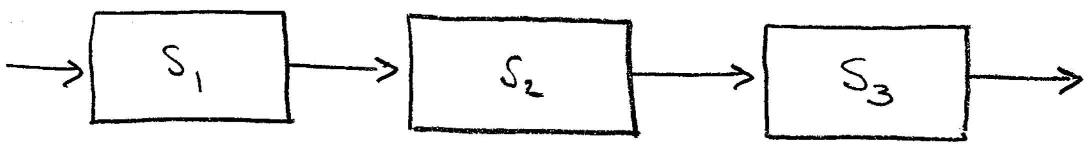
(a) T or F "Series connection of two LTI systems is itself LTI"

True.

(b) T or F "Series connection of two nonlinear systems is nonlinear"

False. Take a memoryless nonlinearity that has an inverse and cascade it with its inverse. The cascade is the identity, which is linear.

(c)



$$y_1[n] = \begin{cases} x_1[n/2] & n \text{ even} \\ 0 & n \text{ odd} \end{cases} \qquad y_3[n] = x_3[2n]$$

$$y_2[n] = x_2[n] + \tfrac{1}{2} x_2[n-1] + \tfrac{1}{4} x_2[n-2]$$

Is Linear? Is Time-invariant.

Linearity clearly holds since subsystems linear.

For time-invariance consider

$$\begin{aligned} y_3[n] = x_3[2n] &= y_2[2n] \\ &= x_2[2n] + \tfrac{1}{2} x_2[2n-1] + \tfrac{1}{4} x_2[2n-2] \\ &= y_1[2n] + \tfrac{1}{2} y_1[2n-1] + \tfrac{1}{4} y_1[2n-2] \\ &= x_1[n] + 0 + \tfrac{1}{4} x_1[n-1]. \end{aligned}$$

$\therefore$ Time inv.

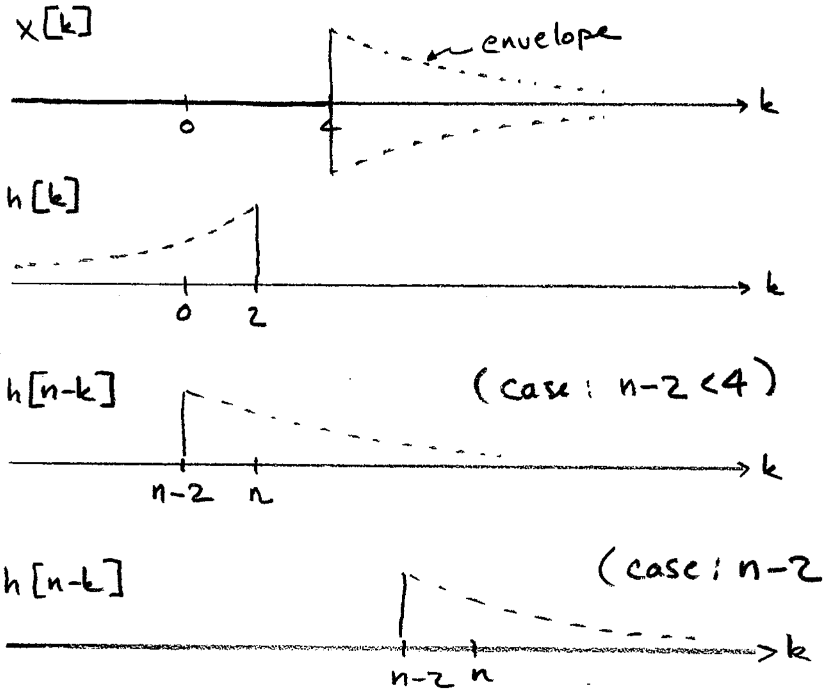
7. <u>O+W 2.21 (c,d)</u> Compute $y[n] = x * h\,[n]$

$$= \sum_k x[k]\,h[n-k]$$

c) $x[n] = \left(-\frac{1}{2}\right)^n u[n-4]$

$h[n] = 4^n\, u[2-n]$.

Draw pictures to find correct limits on sum.



There are 2 cases depending on $n-2 \underset{<}{\overset{\geq}{\text{or}}} 4$

<u>Case</u>: $n-2 < 4 \iff n < 6$

$$y[n] = \sum_{k=4}^{\infty} x[k]\,h[n-k] = \sum_{k=4}^{\infty} \left(-\tfrac{1}{2}\right)^k 4^{n-k}$$

$$= 4^n \sum_{k=4}^{\infty} \left(-\tfrac{1}{8}\right)^k = 4^n \left(-\tfrac{1}{8}\right)^4 \sum_{\ell=0}^{\infty} \left(-\tfrac{1}{8}\right)^{\ell}$$

$$= 4^n \left(\tfrac{1}{8}\right)^4 \frac{1}{1+1/8}$$

$$= \frac{4^n}{8^3 \cdot 9} = \frac{4^n}{4608}$$

Case: $n-2 \geq 4 \iff n \geq 6$

$$y[n] = \sum_{k=n-2}^{\infty} x[k]\, h[n-k] = \sum_{k=n-2}^{\infty} \left(-\tfrac{1}{2}\right)^k 4^{n-k}$$

$$= 4^n \sum_{k=n-2}^{\infty} \left(-\tfrac{1}{8}\right)^k = 4^n \left(-\tfrac{1}{8}\right)^{n-2} \sum_{\ell=0}^{\infty} \left(-\tfrac{1}{8}\right)^\ell$$

$$= \left(-\tfrac{1}{2}\right)^n 8^2 \frac{1}{1+1/8} = \left(-\tfrac{1}{2}\right)^n 8^2 \cdot 8/9$$

$$= \left(-\tfrac{1}{2}\right)^n \frac{512}{9}$$

Together the two cases give

$$y[n] = \begin{cases} \dfrac{4^n}{4608} & n < 6 \\ \left(-\tfrac{1}{2}\right)^n \dfrac{512}{9} & n \geq 6 \end{cases}$$

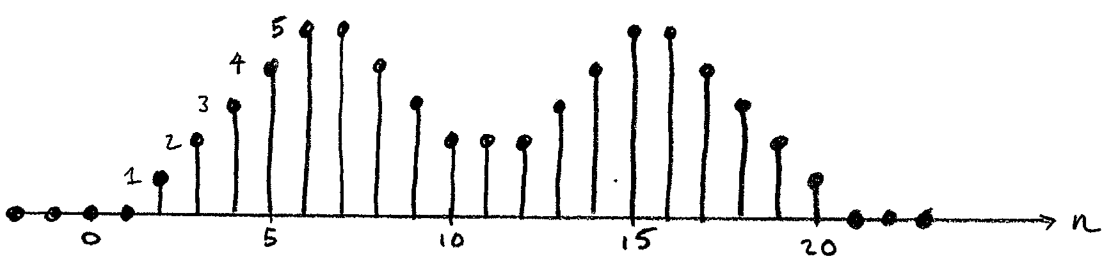
$4096/4608$

d) This one is easier to compute directly

$$y[n] = \sum_k x[k]\, h[n-k]$$

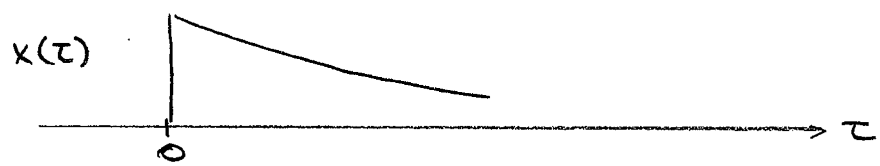
$$= x[0]\,h[n] + x[1]\,h[n-1] + x[2]\,h[n-2] + x[3]\,h[n-3] + x[4]\,h[n-4]$$

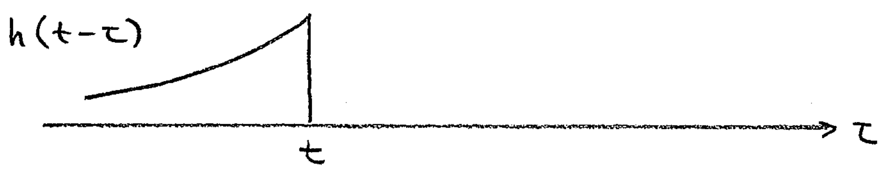
$$= h[n] + h[n-1] + h[n-2] + h[n-3] + h[n-4]$$



8. <u>O+W 2.22 (a,c,e)</u> Find $y(t) = x * h(t)$.
$$= \int x(\tau)h(t-\tau)\,d\tau$$

(a) $x(t) = e^{-\alpha t} u(t)$
$h(t) = e^{-\beta t} u(t)$
(for cases $\alpha \neq \beta$ and $\alpha = \beta$).





Clearly if $t<0$ then $y(t)=0$. So consider $t>0$.

$$y(t) = \int_0^t x(\tau)h(t-\tau)\,d\tau = \int_0^t e^{-\alpha\tau} e^{-\beta(t-\tau)}\,d\tau$$

$$= e^{-\beta t}\int_0^t e^{-(\alpha-\beta)\tau}\,d\tau$$

Case: $\alpha = \beta \implies y(t) = t e^{-\beta t}$ for $t>0$.

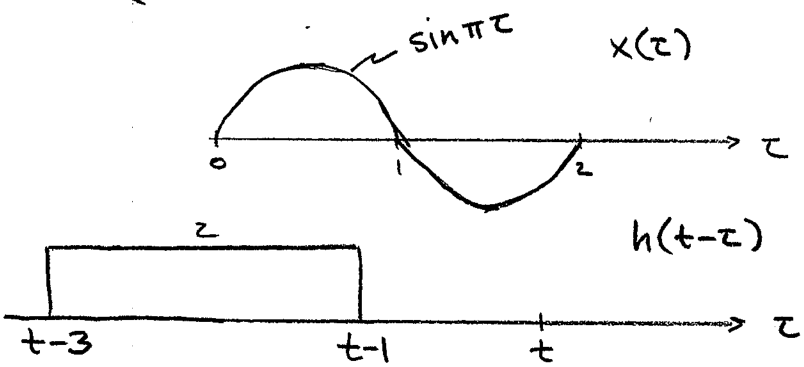
Case: $\alpha \neq \beta$

$$y(t) = e^{-\beta t} \frac{-1}{\alpha-\beta} e^{-(\alpha-\beta)\tau}\Big|_{\tau=0}^{t}$$

$$= -\frac{e^{-\beta t}}{\alpha-\beta}\left[e^{-(\alpha-\beta)t} - 1\right]$$

$$= \frac{e^{-\beta t} - e^{-\alpha t}}{\alpha-\beta} \quad \text{for } t>0.$$

(c)



<u>Easy Cases</u>

$t-1<0 \iff t<1$ then $y(t)=0$

$t-3>2 \iff t>5$ then $y(t)=0$

<u>Case</u> $0<t-1<2 \iff 1<t<3$

$$y(t) = \int_0^{t-1} 2\sin\pi\tau \, d\tau = -\frac{2}{\pi}\cos\pi\tau \Big|_{\tau=0}^{t-1}$$

$$= -\frac{2}{\pi}\left[\cos\pi(t-1) - 1\right] = \frac{2}{\pi}\left[1-\cos\pi(t-1)\right]$$

<u>Case</u> $0<t-3<2 \iff 3<t<5$

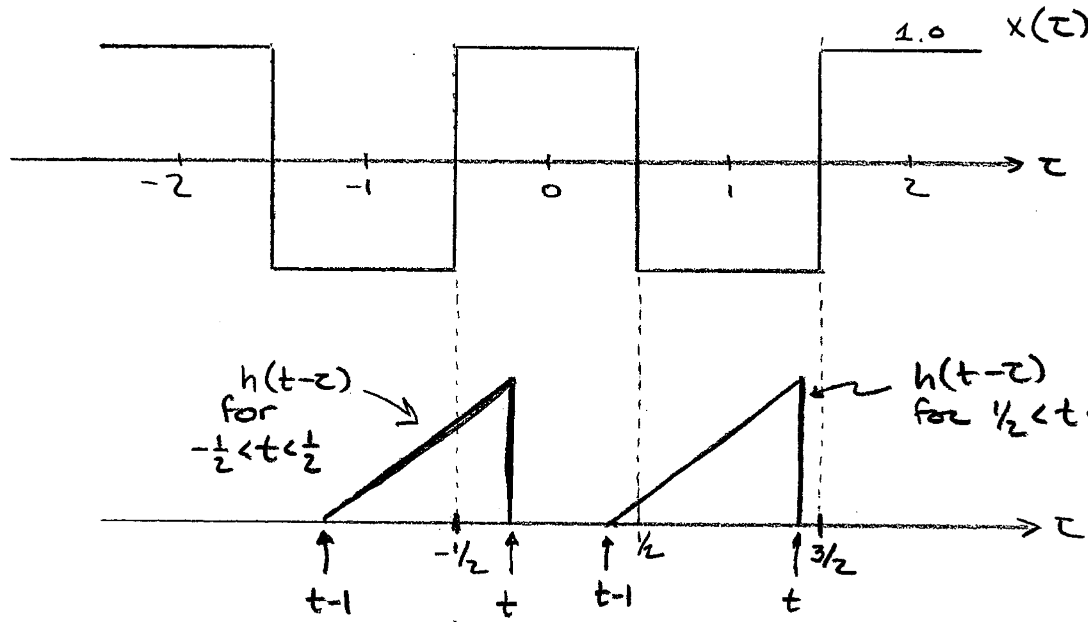
$$y(t) = \int_{t-3}^{2} 2\sin\pi\tau \, d\tau = -\frac{2}{\pi}\left[\cos 2\pi - \cos\pi(t-3)\right]$$

$$= \frac{2}{\pi}\left[\cos\pi(t-3) - 1\right].$$

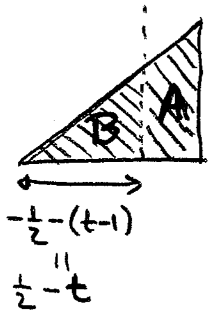
$$y(t) = \begin{cases} 0 & \text{if } t<1 \\ \frac{2}{\pi}\left[1-\cos\pi(t-1)\right] & \text{if } 1<t<3 \\ \frac{2}{\pi}\left[\cos\pi(t-3)-1\right] & \text{if } 3<t<5 \\ 0 & \text{if } t>5 \end{cases}$$

(e) Note: Input periodic of period $=2$ implies output also periodic of period $=2$. So only need compute output over one period.

Lets compute $y(t)$ for $-\frac{1}{2} < t < \frac{3}{2}$



<u>Case</u> $-\frac{1}{2} < t < \frac{1}{2}$



$$y(t) = \underbrace{\text{area}(A)}_{\frac{1}{2} - \text{area}(B)} - \text{area}(B)$$

$$\therefore \; y(t) = \frac{1}{2} - 2\,\text{area}(B)$$

$$= \frac{1}{2} - 2\,\frac{1}{2}\left(\frac{1}{2}-t\right)^2$$

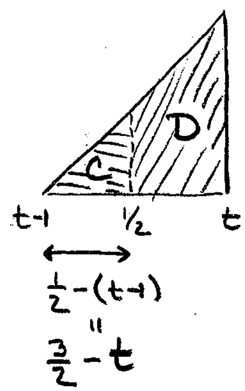
$$= \frac{1}{2}\left[1 - 2\left(\frac{1}{4} - t + t^2\right)\right]$$

$$= \frac{1}{2}\left[1 - \frac{1}{2} + 2t - 2t^2\right]$$

$$= \frac{1}{2}\left[\frac{1}{2} + 2t - 2t^2\right]$$

$$= \frac{1}{4} + t - t^2$$

Case $\frac{1}{2} < t < \frac{3}{2}$



$$y(t) = \text{area}(C) - \underbrace{\text{area}(D)}_{= \frac{1}{2} - \text{area}(C)}$$

$$= 2\,\text{area}(C) - \tfrac{1}{2}$$

$$= 2\,\frac{1}{2}\left(\frac{3}{2}-t\right)^2 - \frac{1}{2}$$

$$= \frac{9}{4} - 3t + t^2 - \frac{1}{2}$$

$$= \frac{7}{4} - 3t + t^2$$

$$\therefore\ y(t) = \begin{cases} \frac{1}{4} + t - t^2 & \text{for } -\frac{1}{2} < t < \frac{1}{2} \\ \frac{7}{4} - 3t + t^2 & \text{for } \frac{1}{2} < t < \frac{3}{2} \\ \text{and periodic for remaining } t. \end{cases}$$