Team Ninja-Sharks
Problem 4
Find the degree of the extension $\mathbb{Q}(\alpha)$ over $\mathbb{Q}$, where

$$\alpha = \sqrt{\frac{1+\sqrt{5}}{2}}$$

Solution:
Consider $f(x) = x^4 - x^2 - 1$. I claim that f is a minimal polynomial for α.
α is a root of f since $f(\alpha) = 0$. So all that remains to show is that f is irreducible over $\mathbb{Q}$. f does not have any linear factors since $f(1) \neq 0$, and $f(-1) \neq 0$. (And thus f has no cubic factors). Suppose f has a quadratic factor. Then for some a, b, c, d,

$$\begin{aligned} x^4 - x^2 - 1 &= (x^2 + ax + b)(x^2 + cx + d) \\ &= x^4 + (a+c)x^3 + (b + ac + d)x^2 + (ad + bc)x + bd \end{aligned}$$

We have

$$a + c = 0, \implies a = -c \quad (1)$$
$$d + ac + b = -1 \quad (2)$$
$$ad + bc = 0 \implies ad = -bc \quad (3)$$
$$bd = -1, \implies b \in \{-1, 1\} \quad (4)$$

- Case 1: $b = 1, \implies d = -1$ from equation 4, $\implies a = c$ from equation 3, but from equation 1, $a = -c \implies a = c = 0$, then from equation 2 we get $0 = -1$. $\rightarrow\leftarrow$
- Case 2: $b = -1$. This is the same case as the first.

Therefore f has no quadratic factors, and f is irreducible over $\mathbb{Q}$. Then f is the minimal polynomial for α, and since the degree of f is 4, the degree of the extension of $\mathbb{Q}(\alpha)$ over $\mathbb{Q}$ is 4.