EE 301 Exam 2 3/4/13

Name: Solution

General Instructions:

- **You have 60 minutes to complete the exam.**
- Write your name on every page of the exam.
- The exam is closed book and closed notes. Calculators are not allowed.
- Your work must be explained to receive full credit. All plots must be carefully drawn with axes labeled.
- Point values for each problem are as indicated. The exam totals 100 points.
- If you finish the exam during the first 50 minutes, you may turn it in and leave. During the last 10 minutes you must remain seated until we pick up exams from everyone.
- Problems labeled with **LO** indicate that the problem is used to determine student satisfaction of course learning objectives.

This exam is for Krogmeier's section of 301.

Do not open the exam until you are told to begin.

Name: JVK

Problem 1. *Computing Basic Fourier Series and Fourier Transform Pairs.* [40 pts. total, **LO-iv**]

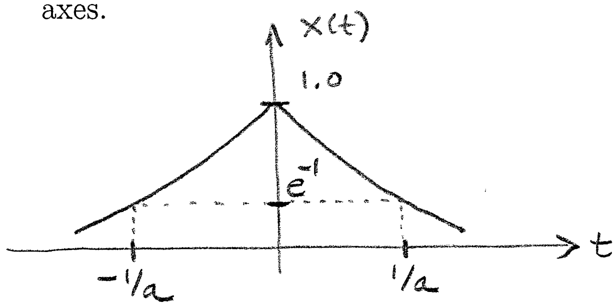
This problem has two unrelated parts.

(a) The purpose of this part is to compute the Fourier Transform of

$$x(t) = e^{-a|t|}, \quad -\infty < t < \infty,$$

$a > 0$, from first principles.

(a-1) [2 pts.] Make a careful sketch of $x(t)$ vs. t labeling important points on both axes.



(a-2) [10 pts.] Substitute $x(t)$ into the forward Fourier Transform integral, evaluate it to compute $X(j\omega)$, and simplify.

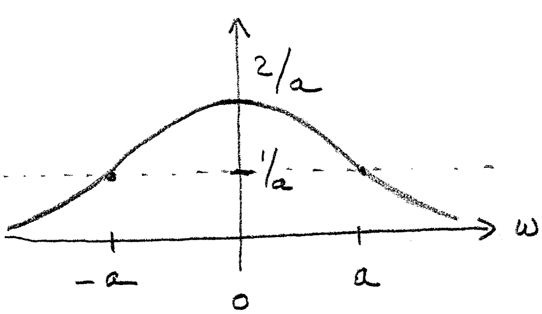
$$\int_{-\infty}^{\infty} e^{-a|t|} e^{-j\omega t}\,dt = \int_{0}^{\infty} \underbrace{e^{-at} e^{-j\omega t}}_{e^{-(a+j\omega)t}}\,dt + \int_{-\infty}^{0} \underbrace{e^{at} e^{-j\omega t}}_{e^{(a-j\omega)t}}\,dt$$

$$= \frac{1}{-(a+j\omega)} e^{-(a+j\omega)t}\Big|_{t=0}^{\infty} + \frac{1}{(a-j\omega)} e^{(a-j\omega)t}\Big|_{t=-\infty}^{0}$$

$$= \frac{1}{a+j\omega} + \frac{1}{a-j\omega} = \frac{a-j\omega + a+j\omega}{(a+j\omega)(a-j\omega)} = \frac{2a}{a^2+\omega^2}$$

(a-3) [3 pts.] Carefully plot $X(j\omega)$ vs. ω labeling important points on both axes.

$$\therefore \; X(j\omega) = \frac{2a}{a^2+\omega^2}$$ $\longrightarrow$ note that it is real-valued so only need one plot



Problem 1. (cont'd.) Name: **JVK**

(b) You've seen the signals below on Exam 1. Now you will look at them in the frequency domain. It is permitted to use the Fourier Transform and Series tables appended to the end of the exam as long as you properly cite the entries or properties you use.

(b-1) [5 pts.] Let $w(t) = e^{-3t}u(t)$ and find the Fourier transform $W(j\omega)$.

$$W(j\omega) = \int_0^\infty e^{-3t} e^{-j\omega t}\, dt = \int_0^\infty e^{-(3+j\omega)t}\, dt = \frac{1}{-(3+j\omega)} e^{-(3+j\omega)t}\Big|_{t=0}^{\infty}$$

$$= \frac{1}{3+j\omega} \quad \text{(or find it in the Table)}$$

(b-2) [3 pts.] Find the value of

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} |W(j\omega)|^2 d\omega = \int_{-\infty}^{\infty} |w(t)|^2 dt \quad \text{(by Parseval)}$$

using Parseval's relation.

In this case the time domain integral is easier to evaluate than the freq. domain integral.

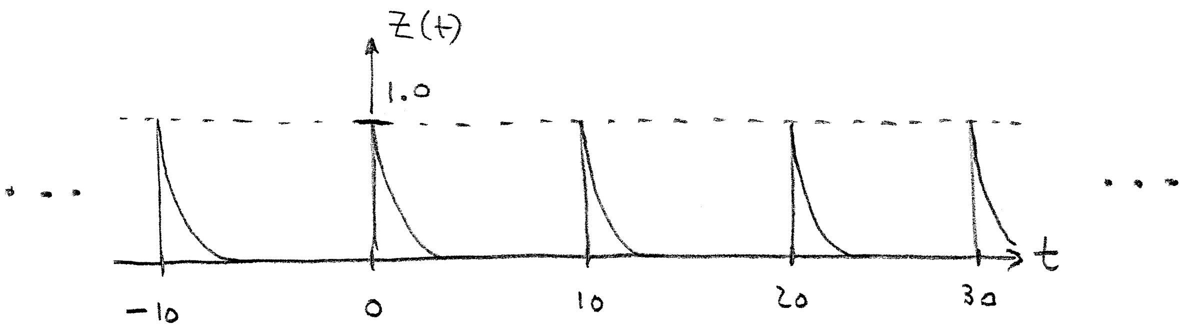
$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|\frac{1}{3+j\omega}\right|^2 d\omega = \int_0^\infty e^{-6t}\, dt = \frac{1}{-6} e^{-6t}\Big|_{t=0}^{\infty} = -\frac{1}{6}(0-1) = \frac{1}{6}$$

$$\therefore \quad \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{d\omega}{9+\omega^2} = \frac{1}{6} \quad \text{(= the energy in the pulse } w(t)\text{)}.$$

(b-3) [2 pts.] Define a periodic signal by

$$z(t) = \sum_{k=-\infty}^{\infty} w(t - 10k).$$

Plot it using good engineering common sense. (e^{-30} is very small)



Problem 1. (cont'd.) Name: JVK

(b) (cont'd.)

(b-4) [10 pts.] Compute the Fourier Series coefficients of $z(t)$ and the Fourier Transform $Z(j\omega)$.

$$Z_k = \frac{1}{10}\int_0^{10} z(t)\, e^{-jk\pi t/5}\, dt \qquad \omega_0 = \frac{2\pi}{T} = \frac{2\pi}{10} = \frac{\pi}{5}$$

$$\approx \frac{1}{10}\int_0^{\infty} e^{-3t} e^{-jk\pi t/5}\, dt = \frac{1}{10}\int_0^{\infty} e^{-(3+jk\pi/5)t}\, dt$$

$$= \frac{1/10}{-(3+jk\pi/5)}\, e^{-(3+jk\pi/5)t}\Big|_{t=0}^{\infty} = \frac{1/10}{3+jk\pi/5}$$

$$\therefore\ Z_k \approx \frac{1/2}{15+jk\pi}$$

approx. comes from observation that e^{-30} is small enough to ignore. Interestingly, this result is actually exact

Then from the first entry in the Fourier Transform pairs table have

$$z(t) = \sum_{k=-\infty}^{\infty}\left(\frac{1/2}{15+jk\pi}\right) e^{jk\pi t/5}$$

$$\updownarrow$$

$$Z(j\omega) = \pi \sum_{k=-\infty}^{\infty}\left(\frac{1}{15+jk\pi}\right)\delta(\omega - k\pi/5)$$

Problem 1. (cont'd.) Name: JVK

(b) (cont'd.)

(b-5) [5 pts.] Using Parseval for Fourier Series derive the identity

$$\frac{1}{15} = \frac{1}{225} + 2\sum_{k=1}^{\infty} \frac{1}{225 + \pi^2 k^2}.$$

From the Table of Fourier Series properties

$$\underbrace{\frac{1}{10}\int_0^{10} z^2(t)\,dt}_{} = \sum_{k=-\infty}^{\infty} |z_k|^2 = |z_0|^2 + 2\sum_{k=1}^{\infty} |z_k|^2$$

$$= \frac{1}{4}\cdot\frac{1}{225} + 2\sum_{k=1}^{\infty} \frac{1/4}{225 + k^2\pi^2}$$

To within a good engineering approximation

$$\frac{1}{10}\int_0^{10} z^2(t)\,dt = \text{average power}$$

$$\approx \frac{\text{energy in } w(t)}{\text{period}} = \frac{1/6}{10} = \frac{1}{60}$$

$$\therefore \frac{1}{60} \approx \frac{1}{4}\cdot\frac{1}{225} + \frac{1}{2}\sum_{k=1}^{\infty} \frac{1}{225 + k^2\pi^2}$$

$$\frac{1}{15} \approx \frac{1}{225} + 2\sum_{k=1}^{\infty} \frac{1}{225 + k^2\pi^2}$$

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Loose Ends on Prob. 1b

this material was not requested of students .... just FYI

Re: $Z_k = \dfrac{1/2}{15 + jk\pi}$ is actually exact

To evaluate $Z_k = \dfrac{1}{10}\displaystyle\int_0^{10} z(t)\, e^{-jk\pi t/5}\, dt$ we need

$z(t)$ for $0 \le t < 10$

Note that for this range of t

$$u(t-10k) = 0 \quad \text{if} \quad k \geq 1$$

$$u(t-10k) = 1 \quad \text{if} \quad k < 1$$

Therefore

$$z(t) = \sum_{k=-\infty}^{0} e^{-3(t-10k)} \qquad \text{for } 0 \le t < 10$$

$$= e^{-3t} \sum_{k=-\infty}^{0} e^{30k} = e^{-3t} \sum_{\ell=0}^{\infty} e^{-30\ell}$$

$$= e^{-3t} \frac{1}{1-e^{-30}} \qquad (*)$$

Now evaluate the Fourier Series coefficients

$$Z_k = \frac{1}{10} \int_0^{10} e^{-3t} \frac{1}{1-e^{-30}}\, e^{-jk\pi t/5}\, dt$$

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$$Z_k = \frac{1}{10}\frac{1}{1-e^{-30}}\int_0^{10} e^{-(3+jk\pi/5)t}\,dt$$

$$= \frac{1}{10}\frac{1}{1-e^{-30}}\frac{1}{-(3+jk\pi/5)}\, e^{-(3+jk\pi/5)t}\Big|_{t=0}^{10}$$

$$= \frac{1}{10}\frac{1}{1-e^{-30}}\frac{1}{3+jk\pi/5}\left[1 - e^{-30}\underbrace{e^{-j2\pi k}}_{1}\right]$$

$$= \frac{1}{10}\frac{1}{3+jk\pi/5} = \frac{1}{10}\frac{5}{15+jk\pi}$$

$$= \frac{1/2}{15+jk\pi}$$

Re: $\frac{1}{10}\int_0^{10} Z^2(t)\,dt \approx \frac{1}{60}$ really is an approx. Hence the "identity" of Prob 1(b-5) is only an approximation

Over the interval $0 \le t < 10$

$$Z^2(t) = e^{-6t}\left(\frac{1}{1-e^{-30}}\right)^2$$ from the result in (*)

Then

$$\frac{1}{10}\int_0^{10} Z^2(t)\,dt = \frac{1}{10}\left(\frac{1}{1-e^{-30}}\right)^2\left(-\frac{1}{6}\right)e^{-6t}\Big|_{t=0}^{10}$$

$$= \frac{1}{60}\left(\frac{1}{1-e^{-30}}\right)^2\left(1-e^{-60}\right)$$

Continuing to simplify

$$\frac{1}{10}\int_0^{10} z^2(t)\,dt = \frac{1}{60}\,\frac{(1-e^{-30})(1+e^{-30})}{(1-e^{-30})^2}$$

$$= \frac{1}{60}\,\frac{1+e^{-30}}{1-e^{-30}}$$

Therefore, the exact identity is

$$\frac{1}{15}\left(\frac{1+e^{-30}}{1-e^{-30}}\right) = \frac{1}{225} + 2\sum_{k=1}^{\infty}\frac{1}{225+k^2\pi^2}$$

Re: Another Way to Compute the Fourier Transform $Z(j\omega)$

$$z(t) = \sum_{k=-\infty}^{\infty} w(t-10k)$$

Take the Fourier Transform term by term and add them up. Note that

$$\mathcal{F}\{w(t-10k)\} = e^{-j\omega 10k}\,W(j\omega)$$

$$\Longrightarrow$$

$$Z(j\omega) = \sum_{k=-\infty}^{\infty} e^{-j\omega 10k}\,W(j\omega)$$

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$$Z(j\omega) = W(j\omega) \sum_{k=-\infty}^{\infty} e^{-j\omega 10 k}$$

Is periodic in ω with period $W = \pi/5$

Therefore, adapting the identity given in class (sometimes called the Poisson Sum Formula)

$$\sum_{k=-\infty}^{\infty} e^{-j\omega 10 k} = \frac{\pi}{5} \sum_{\ell=-\infty}^{\infty} \delta(\omega - \ell\pi/5)$$

Substituting and using $W(j\omega) = \dfrac{1}{3+j\omega}$ we get

$$Z(j\omega) = \left(\frac{1}{3+j\omega}\right) \frac{\pi}{5} \sum_{\ell=-\infty}^{\infty} \delta(\omega - \ell\pi/5)$$

$$= \frac{\pi}{5} \sum_{\ell=-\infty}^{\infty} \left(\frac{1}{3+j\omega}\right) \delta(\omega - \ell\pi/5)$$

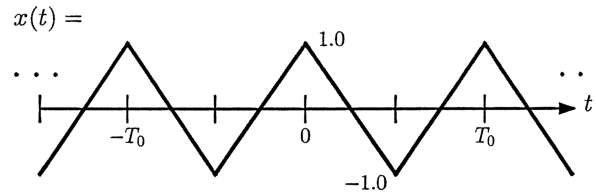
$$= \frac{\pi}{5} \sum_{\ell=-\infty}^{\infty} \left(\frac{1}{3+j\ell\pi/5}\right) \delta(\omega - \ell\pi/5)$$

$$= \pi \sum_{\ell=-\infty}^{\infty} \frac{1}{15+j\pi\ell} \delta(\omega - \ell\pi/5)$$

ie same result as in (b-4).

Name: JVK

Problem 2. *Fourier Transform and Series Properties.* [35 pts. total, **LO-iv**]



$$\longleftrightarrow x_n = \begin{cases} 4/(\pi^2 n^2) & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

(a) Consider the periodic signal $x(t)$ and its Fourier Series coefficients shown above, i.e., the above satisfy

$$x(t) = \sum_{k=-\infty}^{\infty} x_k e^{jk\omega_0 t}; \text{ where } x_n = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t) e^{-jn\omega_0 t} \mathrm{d}t.$$

Answer the following by performing appropriate operations on the defining equations above.

(a-1) [5 pts.] What are the Fourier Series coefficients for $\bar{x}(t) \overset{\text{def}}{=} x(t - T_0/4)$? Show your work.

$$\bar{x}(t) = x(t - T_0/4) = \sum_{k=-\infty}^{\infty} x_k e^{jk\omega_0(t-T_0/4)} = \sum_{k=-\infty}^{\infty} \left(x_k e^{-jk\omega_0 T_0/4}\right) e^{jk\omega_0 t}$$

From uniqueness of Fourier Series representations and the fact that $\omega_0 T_0 = 2\pi$

$$\bar{x}_k = x_k e^{-jk\omega_0 T_0/4} = x_k e^{-jk\pi/2} = (-j)^k x_k \quad k \in \mathbb{Z}$$

$$= \begin{cases} -j4/(\pi^2 k^2) & \text{if } k \text{ odd, of form } k = 4\ell+1 \\ j4/(\pi^2 k^2) & \text{if } k \text{ odd, of form } k = 4\ell+3 \\ 0 & \text{if } k \text{ even} \end{cases}$$

(a-2) [5 pts.] What are the Fourier Series coefficients for $\hat{x}(t) \overset{\text{def}}{=} x(t - T_0/2)$? Show your work.

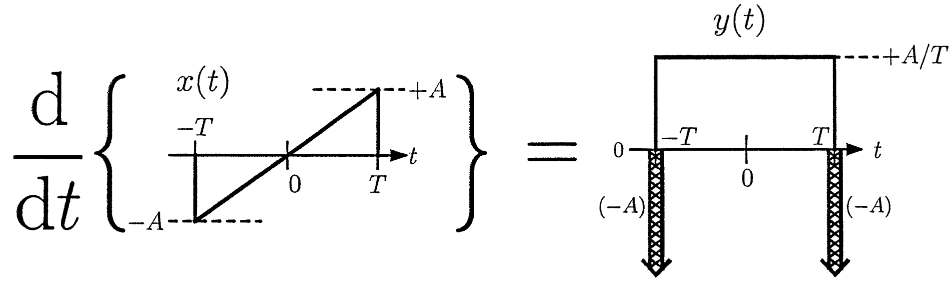
Same method as above results in

$$\hat{x}_k = x_k e^{-j\omega_0 T_0/2} = x_k e^{-jk\pi} = (-1)^k x_k$$

$$= \begin{cases} -4/(\pi^2 k^2) & k \text{ odd} \\ 0 & k \text{ even} \end{cases}$$

Note that for this shift and this signal $\hat{x}(t) = -x(t)$ whence should have $\hat{x}_k = -x_k$

Problem 2. (cont'd.) Name: JVK



(b) The figure above shows a symmetric time-limited ramp $x(t)$ and its derivative $y(t)$, which consists of a symmetric rectangular pulse summed with two Dirac delta functions. From the derivative property of the Fourier Transform we know that

$$j\omega X(j\omega) = Y(j\omega).$$

There are two ways we might compute the Fourier Transform $X(j\omega)$: the direct approach and the sneaky approach.

(b-1) [10 pts.] The direct approach. Plug $x(t)$ into the forward Fourier Transform Integral and evaluate to compute $X(j\omega)$. This will require integration by parts[1]. Simplify.

$$X(j\omega) = \int_{-T}^{T} \frac{A}{T} t\, e^{-j\omega t}\, dt$$

parts: $u = t$, $du = dt$

$dv = e^{-j\omega t} dt$, $v = -\frac{1}{j\omega} e^{-j\omega t}$

$$= \frac{A}{T}\left[-\frac{t}{j\omega} e^{-j\omega t} \Big|_{t=-T}^{T} - \int_{-T}^{T} \left(-\frac{1}{j\omega}\right) e^{-j\omega t}\, dt \right]$$

$$= -\frac{A}{j\omega T}\left[T e^{-j\omega T} + T e^{+j\omega T} \right] - \frac{A}{T}\left(-\frac{1}{j\omega}\right)^2 e^{-j\omega t} \Big|_{t=-T}^{T}$$

$$= j\frac{2A}{\omega} \cos \omega T + \frac{A}{T}\frac{1}{\omega^2} \left(\frac{e^{-j\omega T} - e^{+j\omega T}}{-j2} \right)(-j2)$$

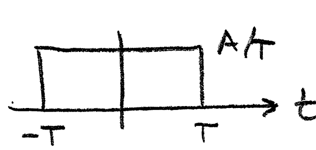
$$= j\frac{2A}{\omega} \cos \omega T - j\frac{2A}{\omega^2 T} \sin \omega T$$

[1]Recall:
$$\int u dv = uv - \int v du.$$

Problem 2. (cont'd.) Name: JVK

(b) (cont'd.)

(b-2) [10 pts.] <u>The sneaky approach.</u> Notice that $Y(j\omega)$ can be found as the sum of three Fourier Transforms from the Transform Table. Then solve for $X(j\omega)$ using the derivative property mentioned above. Simplify.

$$Y(j\omega) = \mathcal{F}\Big\{$$ 

$$\Big\} + \mathcal{F}\{-A\,\delta(t+T)\} + \mathcal{F}\{-A\,\delta(t-T)\}$$

$$= \frac{2A}{T\omega}\sin\omega T - A e^{j\omega T} - A e^{-j\omega T}$$

$$= \frac{2A}{T\omega}\sin\omega T - 2A\cos\omega T$$

$$\therefore \quad j\omega X(j\omega) = Y(j\omega)$$

$$X(j\omega) = \frac{\frac{2A}{T\omega}\sin\omega T - 2A\cos\omega T}{j\omega}$$

$$= j\frac{2A}{\omega}\cos\omega T - j\frac{2A}{T\omega^2}\sin\omega T$$

(agrees with direct approach)

Problem 2. (cont'd.) Name: JVK

(b) (cont'd.)

(b-3) [5 pts.] A worrisome note. What happens if we try the sneaky approach on the unit step function $u(t) \leftrightarrow U(j\omega)$? Knowing that the derivative of the unit step is equal to the Dirac delta function and applying the derivative property we get

$$j\omega U(j\omega) = 1.$$

Compare with the unit step's Fourier Transform from the attached table. What has happened? Conjecture a condition that could be required of $x(t)$ so that the sneaky approach will work. Explain your thinking.

Applying to unit step would get $1/j\omega$ for the FT of $u(t)$. But Table:

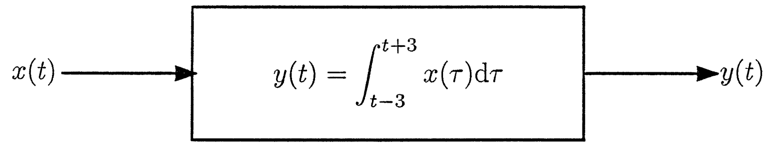
$$U(j\omega) = \frac{1}{j\omega} + \pi\delta(\omega)$$

which also satisfies $j\omega U(j\omega) = 1$.

Conjecture: Sneaky approach works for finite energy pulses.

Name: JVK

Problem 3. *LTI Systems and Fourier Transforms.* [25 pts. total, **LO-v**]



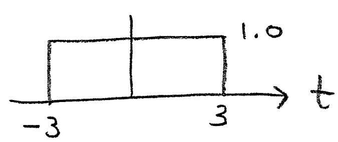
For the system above

$$Y(j\omega) = H(j\omega)X(j\omega).$$

Find $H(j\omega)$ and carefully plot it below. Explain your work.

System is LTI with impulse response

$h(t) =$



Hence transfer function $H(j\omega) \longleftrightarrow h(t)$. From the Table:

$$H(j\omega) = \frac{2\sin 3\omega}{\omega} = 6 \cdot \frac{\sin 3\omega}{3\omega}$$

sinc with zero crossings at

$3\omega = \pi k$

$\omega = \frac{\pi}{3}k \quad k \neq 0$