Text Example 4.1

$$X(\omega)$$

$$\boxed{\underset{x(t)}{e^{-at}u(t)} \xleftrightarrow{\mathcal{F}} \frac{1}{a+j\omega}} \quad \left(= \frac{1}{a} \cdot \frac{1}{1+j\frac{\omega}{a}} \right)$$

$$|X(\omega)| = \frac{1}{\sqrt{a^2+\omega^2}}, \qquad \angle X(\omega) = -\tan^{-1}\left(\frac{\omega}{a}\right)$$

(See Fig 4.5)

Text Example 4.2

$$y(t) = x(t) + x(-t) \quad \text{where} \quad x(t) = e^{-at}u(t)$$

$$= e^{-at}u(t) + e^{at}u(-t) = e^{-a|t|}$$

$$Y(\omega) = X(\omega) + \cancel{\frac{1}{|-1|}} X(-\omega) \quad \text{since} \quad \text{“} x(at) \xleftrightarrow{\mathcal{F}} \frac{1}{|a|} X\left(\frac{\omega}{a}\right) \text{”}$$

$$= \frac{1}{a+j\omega} + \frac{1}{a-j\omega} = \frac{a-\cancel{j\omega} + a+\cancel{j\omega}}{a^2+\omega^2}$$

$$= \frac{2a}{a^2+\omega^2} \qquad \boxed{e^{-a\cancel{|t|}\,\cancel{u(t)}} \xleftrightarrow{\mathcal{F}} \frac{2a}{a^2+\omega^2}}$$

$$\downarrow$$

$$e^{-a|t|}$$

(Review)

$$x(t) = \delta(t) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega) = 1$$

$$\mathrm{rect}\left(\frac{t}{T}\right) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega) = \frac{\sin\left(T\frac{\omega}{2}\right)}{\frac{\omega}{2}}$$

$$\frac{\sin(Wt)}{\pi t} \overset{\mathcal{F}}{\longleftrightarrow} X(\omega) = \mathrm{rect}\left(\frac{\omega}{2W}\right)$$

$$e^{j\omega_0 t} \overset{\mathcal{F}}{\longleftrightarrow} 2\pi\,\delta(\omega - \omega_0)$$

Example 4.6

If $x(t) = x(t+T)$, $\forall t$ then

$$\underline{x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j\frac{2\pi k}{T}t}} \overset{\mathcal{F}}{\longleftrightarrow} X(\omega) = ?$$

since FT is a linear operator, $\quad \underline{= \sum_{k=-\infty}^{\infty} a_k\, 2\pi\, \delta\left(\omega - k\frac{2\pi}{T}\right)}$

<u>Example 4.8</u> → very important for sampling theory in Chap 7.

$$x(t) = \sum_{k=-\infty}^{\infty} \delta(t-kT)$$: train of delta func. equi-spaced by T.

$$= \sum_{k=-\infty}^{\infty} \underline{\frac{1}{T}}\, e^{jk\frac{2\pi}{T}t} \xleftrightarrow{\mathcal{F}} \sum_{k=-\infty}^{\infty} \frac{2\pi}{T}\, \delta\left(\omega - k\frac{2\pi}{T}\right)$$

$a_k = \frac{1}{T}$

Thus $$\boxed{\sum_{k=-\infty}^{\infty} \delta(t-kT) \xleftrightarrow{\mathcal{F}} \sum_{k=-\infty}^{\infty} \frac{2\pi}{T}\, \delta\left(\omega - k\frac{2\pi}{T}\right)}$$

(See Fig 4.14)

<Review>

- for $x(t)$: real-valued,

$$X(-\omega) = X^*(\omega)$$

or $\underline{X^*(-\omega) = X(\omega)}$

Expressing $X(\omega)$ in polar form :

$$X(\omega) = |X(\omega)| e^{j\angle X(\omega)}$$

This says: $X^*(-\omega) = |X(-\omega)| e^{-j\angle X(-\omega)}$

$$= X(\omega) = |X(\omega)| e^{j\angle X(\omega)}$$

Hence if $x(t)$ is real-valued,

$$\begin{cases} |X(-\omega)| = |X(\omega)| & \text{: magnitude is even func. of freq.} \\ \angle X(-\omega) = -\angle X(\omega) & \text{: phase is odd func. of freq.} \end{cases}$$

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- We proved in class

$$x(at) \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$$

If $a=-1$

$$\boxed{x(-t) \xleftrightarrow{\mathcal{F}} X(-\omega)}$$

Now, if $x(t)$ is real-valued, $X(-\omega) = X^*(\omega)$

Thus, for

$$\boxed{\begin{array}{l} x(t): \text{real-valued.} \\ x(-t) \xleftrightarrow[\mathcal{F}]{} X^*(\omega) \end{array}}$$

- $x_e(t) = \frac{1}{2}\{x(t) + x(-t)\}$: even part

$$X_e(\omega) = \frac{1}{2}\{X(\omega) + X^*(\omega)\} = \text{Real}\{X(\omega)\}$$

$$\boxed{x_e(t) = \mathcal{E}v\{x(t)\} \xleftrightarrow{\mathcal{F}} \text{Real}\{X(\omega)\}}$$

$x_o(t) = \frac{1}{2}\{x(t) - x(-t)\}$: odd part

$$X_o(\omega) = \frac{1}{2}\{X(\omega) - X^*(\omega)\} = j\,\text{Imag}\{X(\omega)\}$$

$$\boxed{x_o(t) = \mathcal{O}d\{x(t)\} \xleftrightarrow{\mathcal{F}} j\,\text{Imag}\{X(\omega)\}}$$

Some properties following from freq. shift property.
= modulation

- We proved in class

$$\boxed{x(t)\, e^{j\omega_0 t} \overset{\mathcal{F}}{\longleftrightarrow} X(\omega-\omega_0)}$$

since $\cos(\omega_0 t) = \frac{1}{2} e^{j\omega_0 t} + \frac{1}{2} e^{-j\omega_0 t}$.

$$\boxed{x(t)\cos(\omega_0 t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{2} X(\omega-\omega_0) + \frac{1}{2} X(\omega+\omega_0)}$$

since $\sin(\omega_0 t) = \frac{1}{2j} e^{j\omega_0 t} - \frac{1}{2j} e^{-j\omega_0 t}$.

$$\boxed{x(t)\sin(\omega_0 t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{2j} X(\omega-\omega_0) - \frac{1}{2j} X(\omega+\omega_0)}$$

Prob 4.21 (e)

$$x(t) = t\, e^{-2t} \sin(4t)\, u(t) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega) = ?$$

Rewrite as $x(t) = t\,[e^{-2t} u(t) \sin(4t)]$

$$= t\, z(t)$$

First, find $Z(\omega)$

$$e^{-2t}u(t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{2+j\omega}$$

$$Z(\omega) = \mathcal{F}\{e^{-2t}u(t)\sin(4t)\}$$

$$= \frac{1}{2j}\frac{1}{2+j(\omega-4)} - \frac{1}{2j}\frac{1}{2+j(\omega+4)}$$

Recall " $t\,z(t) \overset{\mathcal{F}}{\longleftrightarrow} j\frac{d}{d\omega}\{Z(\omega)\}$ "

$\underline{x(t) = t\,z(t)}$ $$X(\omega) = j\frac{d}{d\omega}\{Z(\omega)\}$$

$$= \frac{1}{2}\frac{d}{d\omega}\left\{\frac{1}{2+j(\omega-4)}\right\} + \frac{1}{2}\frac{d}{d\omega}\left\{\frac{1}{2+j(\omega+4)}\right\}$$

$$= \frac{1}{2}\frac{-j}{(2+j(\omega-4))^2} + \frac{1}{2}\frac{-j}{(2+j(\omega+4))^2}$$

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$x(t) \xleftrightarrow{\mathcal{F}} X(\omega)$$

(time-domain) $\quad x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega \quad$ (freq-domain)

- Initial Value Theorem :

$$x(t)\big|_{t=0} = x(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, \cancel{e^{j0}}^{\,1}\, d\omega$$

$$x(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, d\omega$$

Similarly,

$$X(\omega)\big|_{\omega=0} = X(0) = \int_{-\infty}^{\infty} x(t)\, \cancel{e^{-j0}}^{\,1}\, dt$$

$$X(0) = \int_{-\infty}^{\infty} x(t)\, dt$$

- Duality Property if $x(t) \xleftrightarrow{\mathcal{F}} X(\omega)$

then $X(t) \xleftrightarrow[\mathcal{F}]{} 2\pi\, x(-\omega)$