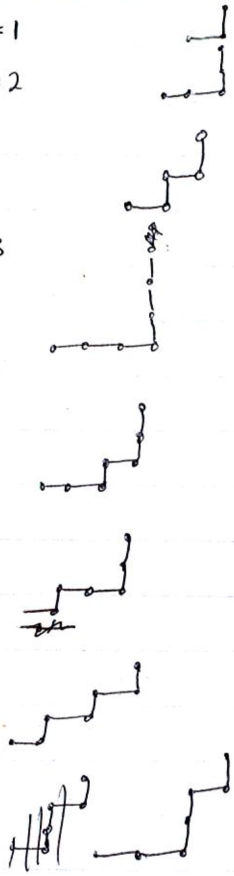
23 FEBRUARY 2012

Challenge:

continuing from last ~~chapter~~ notes:



	walks	multiplication strategy	nested bracket patter
$n=1$		(* *)	( )
$n=2$		((* *) *)	(( ))
		(* (* *))	( )( )
$n=3$		(((* *) *) *)	((( )))
		((* (* *)) *)	(( )( ))
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		((* *)(* *))	(( )) ( )

→ Find recipe

(a) walks ⟷ bracket pattern

(b) multiplication strategy → walks

(c) show that (b) is bijective

my findings:

nested bracket: (= east

) = north

Recall: $a_n = 2a_{n-1} - a_{n-2}$

solving $\lambda_{1,2} = 1$

$\Rightarrow$ a_n = constant. But;

initial cond. $a_1 = 0$, $a_2 = 1$] NOT constant

Note, in diff eq:

$$y^n = C_{n-1} y^{n-1} + \dots + C_1 y + C_0 y$$

→ characteristic equation:

$$\lambda^n = C_{n-1} \lambda^{n-1} + \dots + C_1 \lambda + C_0$$

having $\lambda_1, \dots, \lambda_n$ roots

solution looks like

Thm: $y = \Sigma(b_i e^{\lambda_i t})$

only if λ_i are distinct

(similarly in Lin Alg, $\lambda = \dim(A)$ then diagonalizable)

"Better Theorem" that will explain our discrepency.

Let $\overset{(*)}{a_n} = C_{n-1} a_{n-1} + \dots + C_{n-k} a_{n-k}$

be a homogeneous linear recurrence of order k.

Let $\lambda_1, \dots, \lambda_d$ be distinct characteristic roots.

Let $e_1, \dots, e_d$ be multiplicities of $\lambda_1, \dots, \lambda_d$

↳ i.e. $\lambda^k - C_{n-1}\lambda^{k-1} - \dots - C_1 \lambda - C_0 = (\lambda - \lambda_1)^{e_1} \cdot (\lambda - \lambda_2)^{e_2} \dots (\lambda - \lambda_d)^{e_d}$

note $e_1 + e_2 + \dots e_d = k$

Then, the general solution to (*) is given by.

$$\begin{aligned} a_n = {} & \lambda_1^n (b_{0,1} + b_{1,1} n + b_{2,1} n^2 + \dots + b_{e-1,1} n^{e_1 - 1}) \\ & + \lambda_2^n (b_{0,2} + b_{1,2} n + b_{2,2} n^2 + \dots + b_{e-1,2} n^{e_2 - 2}) \\ & + \vdots \\ & + \lambda_d^n (b_{0,d} + b_{1,d} n + b_{2,d} n^2 + \dots + b_{e-1,d} e^{e-d}) \end{aligned}$$

instead of trying to memorize the equation, keep in mind:

(1) every distinct root contributes to the solution

(2) each ________, ____________ has a contributing polynomial.

e.g.

$$a_n = 2a_{n-1} - a_{n-2}$$

then,

$\lambda_1 = 1$ and $a_n = 1^n(b_0 + b_1 n)$

$e_1 = 2$ $\quad = b_0 + b_1 n$

→ given initial conditions $a_1 = 1$ and $a_2 = 2$

$\Rightarrow a_1 = 1 = b_0 + b_1(1)$

$a_2 = 2 = b_0 + b_1(2)$

$\Rightarrow b_1 = 1 \quad b_0 = 0 \qquad a_n = n$ ✓

e.g. 2

$$a_n = 10a_{n-1} - 33a_{n-2} + 36a_{n-3}$$

→ characteristic eq.

$$\lambda^3 = 10\lambda^2 - 33\lambda + 36$$

⇒ it is difficult to find roots of degree ≥ 3 and impossible to find formula for finding roots for degree > 4.

Galois Theory : measures complexity of eqs and polynomial degree > 4 is "too complex ?"

⇒ More on MA 450 or 453.

For degree = 3

<u>Thm</u>

If polynomial $p(x)$ has integer coefficient and the leading coefficient is 1, then all nice (integer) roots are the divisors of the constant term..

In our case, constant term is 36 and we can test..

$\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 9, \pm 12, \pm 18, \pm 36$

for possible roots.

we find :

$\lambda_1 = 3 \qquad e_1 = 2$

$\lambda_2 = 4 \qquad e_2 = 1$

Then, by theorem

$$a_n = 3^n (b_0 + b_1 n) + 4^n (b_2)$$

given intial condition $a_0 = 1$ $a_1 = 2$ $a_2 = 0$;

(1) $a_0 = 1 = 3^0 (b_0 + b_1(0)) + 4^0 b_2 = b_0 + b_2$

$a_1 = 2 = 3^1 (b_0 + b_1(1)) + 4^1(b_2) = 3b_0 + 3b_1 + 4b_2$

$a_2 = 0 = 3^2 (b_0 + 2b_1) + 4^2(b_2) = 9b_0 + 18b_1 + 16b_2$

in matrix rep.

$\Longrightarrow$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 3 & 3 & 4 & 2 \\ 9 & 8 & 16 & 0 \end{array}\right] \xrightarrow{rref} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 3 & 0 & 2 \\ 0 & 0 & 1 & -3 \end{array}\right]$$

thus, $b_1 = 4$ $b_2 = \frac{2}{3}$ $b_3 = -3$

$$\Rightarrow \quad a_n = 4 \cdot 3^n + \frac{2}{3} \cdot n \cdot 3^n + 4^n (-3)$$

<u>The Inhomogeneous Case</u>

<u>Ex</u> $a_n = a_{n-1} + 1$

Idea: First solve the homogeneous case ("corresponding homogeneous" equation). and then ~~solve~~ Cook up one special solution to the inhomogeneous case.

Homogenous: $a_n = a_{n-1}$

characteric Eq: $\lambda = 1$ and $\lambda_1 = 1$ $e_1 = 1$

and

$$a_n = 1^n (b_0) = 1^n b_0$$

$\leftarrow$ first approx of $a_n = a_{n-1} + 1$.

Inhomogeneus.

By inspection of $a_n = a_{n-1} + 1 \ldots$ same as counting. or $\underline{a_n = n.}$

so we have:

$a_n^{(p)} = a_{n-1}^{p} + 1$ (particular)

$a_n^{(h)} = a_{n-1}^{h}$ (homogeneous)

$(a_n^p + a_n^h) = (a_{n-1}^p + a_{n-1}^h) + 1$ — has a free parameter — take up initial condition. = fits the