Problem 1

$x(t)$ periodic with fundamental period $T$ and fundamental frequency $\omega_0 = 2\pi/T$.

$$x_n = \frac{1}{T} \int x(t) e^{-j\omega_0 t} dt$$

$$x(t) = \sum_n x_n e^{j\omega_0 t}$$

(a) $\overline{x}(t) \triangleq x(t-T/2)$ Clearly $\overline{x}(t)$ is also periodic with $T$ and $\omega_0$. Have

$$\overline{x}_n = \frac{1}{T} \int \overline{x}(t) e^{-j\omega_0 t} dt = \frac{1}{T} \int x(t-T/2) e^{-j\omega_0 t} dt$$

C.o.v. Let $t' = t - T/2$, $dt' = dt$

$$\overline{x}_n = \frac{1}{T} \int x(t') e^{-j\omega_0(t'+T/2)} dt' = \frac{1}{T} e^{-j\omega_0 T/2} \int x(t) e^{-j\omega_0 t} dt$$

$$= e^{-j\omega_0 T/2} x_n = e^{-j\pi} x_n$$

$$\overline{x}_n = (-1)^n x_n$$

(b) $f_n = \left\{ \begin{array}{ll} x_n & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{array} \right.$

From the above can see that $f_n = \frac{x_n + \overline{x}_n}{2}$. From linearity of Fourier series and uniqueness of Fourier series representations we have

$$f(t) = \sum_n f_n e^{j\omega_0 t} = \frac{x(t) + \overline{x}(t)}{2}$$

$$= \frac{x(t) + x(t-T/2)}{2}$$
Problem 1 (cont'd)

Note that \( f(t) \) is periodic with period \( T/2 \) corresponding to a new fundamental frequency of \( \omega_0 = \frac{2\pi}{T/2} = \frac{4\pi}{T} = 2\omega_0 \).

Since the odd-indexed Fourier coefficients are equal to zero can also write

\[
\tilde{f}(t) = \sum_{n=-\infty}^{\infty} f_n e^{i\omega_0 t} = \sum_{k=-\infty}^{\infty} \tilde{f}_{2k} e^{j2\omega_0 t} = \sum_{k=-\infty}^{\infty} \tilde{f}_{2k} e^{jk\omega_0 (2t)} = \tilde{f}(2t)
\]

i.e. \( \tilde{f}(t) = f(t/2) \) i.e. the two expressions are related by time scaling.

\[\uparrow\]

same as \( f(\cdot) \)

but with time expanded

st. \( \tilde{f}(\cdot) \) still has fund.

period = \( T \).

\[\tilde{f}(t) = \frac{x\left(t/2\right) + x\left(t-T/2\right)}{2}\]

(c) Using the result of part (a) we have

\[g_n = \frac{x_n - \bar{x}_n}{2}\]

\[\Rightarrow g(t) = \frac{x(t) - \bar{x}(t)}{2} = \frac{x(t) - x(t-T/2)}{2}\]

Note: A typo was made in definition of \( \bar{g}(t) \). See next page.
(c) (cont'd.)

For part ii: there was a typo - graphical error. Since the \( g_n = 0 \) for \( n \) odd we must have

\[
\hat{g}(t) = \sum_{n} g_{2n} e^{j n \omega_0 t} = 0 \quad \forall t \text{ since } g_{2n} = 0.
\]

This is not what was intended, but it is correct given the typo.

What was intended

\[
\hat{g}(t) = \sum_{n=-\infty}^{\infty} g_{2n+1} e^{j n \omega_0 t}
\]

Start with

\[
g(t) = \sum_{n=-\infty}^{\infty} g_n e^{j n \omega_0 t} = \sum_{n=-\infty}^{\infty} g_n e^{j n \omega_0 t} \quad \text{odd}
\]

\[
= \sum_{k=-\infty}^{\infty} g_{2k+1} e^{j (2k+1) \omega_0 t}
\]

\[
= e^{j \omega_0 t} \sum_{k=-\infty}^{\infty} g_{2k+1} e^{j k \omega_0 (zt)} = e^{j \omega_0 t} \hat{g}(zt).
\]

\[
\therefore \hat{g}(zt) = e^{-j \omega_0 t} g(t)
\]

\[
\hat{g}(t) = e^{-j \omega_0 t/2} g(t/2)
\]

& a phase term.

(In the plot on next page we show

\[
e^{j \omega_0 t/2} \hat{g}(t) = g(t/2) = x(t/2) - x(t - \tau/2)\)
(a) Find the difference equation

\[ y[n] = \frac{1}{2} y[n-1] + x[n] \]

(b) Find the causal impulse response of the above system.

Here replace \( x[n] = \delta[n] \) and assume system at rest at \( n = -1 \) i.e. \( y[-1] = 0 \). Solution to the equation will be \( h[n] \).

\[ \therefore \text{homog. eqn } n > 0 \quad h[n] - \frac{1}{2} h[n-1] = 0 \]

The characteristic equation is

\[ z - \frac{1}{2} = 0 \]

\[ \therefore h[n] = k \left( \frac{1}{2} \right)^n \quad n > 0 \]

Find the proper initial conditions

\[ h[0] = \delta[0] = 1 \]
\[ h[1] = \frac{1}{2} h[0] = \frac{1}{2} \]

\[ \therefore k = 1 \]

\[ h[n] = \left( \frac{1}{2} \right)^n u[n] \]
(c) The impulse response for this new system is clearly

\[ h_{\text{new}}[n] = a h[n] + b h[n-1] \]

\[ = a \left( \frac{1}{2} \right)^n u[n] + b \left( \frac{1}{2} \right)^{n-1} u[n-1] \]

(d) \[ h_{\text{new}}[n] = \begin{cases} 
0 & n < 0 \\
a & n = 0 \\
a \left( \frac{1}{2} \right)^n + b \left( \frac{1}{2} \right)^{n-1} & n \geq 1 
\end{cases} \]

Thus need \( a = 1 \) and

\[ a \left( \frac{1}{2} \right)^n + b \left( \frac{1}{2} \right)^{n-1} = 0 \quad \text{for} \quad n \geq 1 \]

This is if and only if

\( \left( \frac{1}{2} \right)^n \left\{ a + 2b \right\} = 0 \quad \text{for} \quad n \geq 1. \)

\( \Rightarrow \)

\( a + 2b = 0 \quad \Rightarrow \quad b = -a/2 = -\frac{1}{2} \)

(e) The inverse system has impulse response

\[ h_{\text{inverse}}[n] = \begin{cases} 
1 & n = 0 \\
-\frac{1}{2} & n = 1 \\
0 & \text{otherwise} 
\end{cases} \]
\[
\frac{dy}{dt} + 5y = \int_{-\infty}^{\infty} x(\tau) Z(t-\tau) \, d\tau - 2x(t)
\]

\(x(t) = \text{input}\)
\(y(t) = \text{output}\)
\(Z(t) = \text{impulse resp. of system}\)

(b) Taking Fourier transform of the defining integro-diff. equation and recognizing the integral as a convolution

\[j\omega Y(j\omega) + 5Y(j\omega) = Z(j\omega)X(j\omega) - 2X(j\omega)\]

\[(j\omega + 5)Y(j\omega) = (Z(j\omega) - 2)X(j\omega)\]

\[H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{Z(j\omega) - 2}{j\omega + 5}\]

(c) \[Z(t) = e^{-2t} u(t) + \delta(t)\]

(c-i) From transform table and linearity we have

\[Z(j\omega) = \frac{1}{j\omega + 2} + 1\]

(c-ii) Plug into the expression for \(H(j\omega)\) and simplify

\[H(j\omega) = \frac{\frac{1}{j\omega + 2} - 1}{j\omega + 5} = \frac{1 - j\omega - 2}{(j\omega + 5)(j\omega + 2)}\]

\[= \frac{-j\omega - 1}{(j\omega + 5)(j\omega + 2)}\]
(c - iii) Because $H(j\omega)$ is a rational function in $j\omega$, the overall input-output system is modeled by an ordinary finite-order differential equation.

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{-j\omega - 1}{(j\omega)^2 + 7(j\omega) + 10}.$$ 

$$\left((j\omega)^2 Y(j\omega) + 7j\omega Y(j\omega) + 10 Y(j\omega) = -j\omega X(j\omega) - X(j\omega)\right)$$

$$\frac{d^2}{dt^2} y(t) + 7 \frac{dy(t)}{dt} + 10 y(t) = -\frac{1}{dt} x(t) - x(t)$$

(c - iv) With $\eta = j\omega$

$$H(\eta) = \frac{-\eta - 1}{(\eta + 5)(\eta + 2)} = \frac{A}{\eta + 5} + \frac{B}{\eta + 2}$$

$$A = (\eta + 5) H(\eta) \bigg|_{\eta = -5} = \frac{5 - 1}{-3} = -\frac{4}{3}$$

$$B = (\eta + 2) H(\eta) \bigg|_{\eta = -2} = \frac{1}{3}$$

: From Table

$$h(t) = \frac{1}{3} e^{-\frac{2t}{3}} u(t) - \frac{4}{3} e^{-\frac{5t}{3}} u(t)$$

(d) Now

```
   1
  / \  \\
/  \  \\
0---1---t
```

(d - i) From the Fourier transform table

```
 10
-1
```

$\leftrightarrow$ $\frac{2 \sin \omega T_1}{\omega}$
Thus
\[ z(t) \leftrightarrow \frac{2 \sin \omega t}{\omega} \]

But our \( z(t) \) is the above time domain signal delayed by \( \frac{1}{2} \) sec. From the time-shift property
\[ z(t) \leftrightarrow e^{-j\frac{1}{2} \omega} \frac{2 \sin \omega t}{\omega} = z(j\omega) \]

(d-ii) Plugging the expression into the expression for \( H(j\omega) \) get
\[ H(j\omega) = \frac{2e^{-j\frac{1}{2} \omega} \sin \omega t}{j\omega + 5} - \frac{2}{j\omega + 5} \]
\[ = \frac{2e^{-j\frac{1}{2} \omega} \sin \omega t - 2\omega}{\omega(j\omega + 5)} \]

(d-iii) This is not a rational function of \( j\omega \) and so the input-output system does not have a finite order differential equation model.

(d-iv) Can't use our usual method to find the overall impulse response. However, a time domain solution can be used in this case. Can redraw the block diagram:
Can find the impulse response of the last part:

\[ \frac{d}{dt} y(t) = -5y(t) + w(t) \]

The impulse response is found to be:

\[ h_2(t) = e^{-st} u(t). \]

Then the overall impulse response can be found from time domain convolution:

\[ h(t) = h_1 * h_2(t) \]

\[ = (z - 2\delta) * h_2(t) \]

\[ = z * h_2(t) - 2h_2(t) \]

This part we compute directly:

\[ z * h_2(t) = \int_{-\infty}^{\infty} z(t-\tau) h_2(\tau) d\tau \]
From the picture there are 3 cases to consider in the computation of $z * h_2(t)$.

**Case:** $t < 0 \implies z * h_2(t) = 0$

**Case:** $0 < t < 1 \implies$

$$z * h_2(t) = \int_0^t e^{-st} \, dt = \frac{1}{s} e^{-st} \bigg|_0^t = -\frac{1}{s} (e^{-st} - 1) = 1 - \frac{e^{-st}}{5}$$

**Case:** $t > 1 \implies t$

$$z * h_2(t) = \int_{t-1}^t e^{-st} \, dt = -\frac{1}{s} e^{-st} \bigg|_{t-1}^t = -\frac{1}{s} (e^{-st} - e^5 e^{-st}) = e^{-st} \frac{e^5 - 1}{5}$$

Putting the pieces back together:

$$h(t) = \begin{cases} 0 & t < 0 \\ \frac{1 - e^{-st}}{5} - 2e^{-st} & 0 < t < 1 \\ (\frac{e^5 - 1}{5}) e^{-st} - 2e^{-st} & t > 1 \end{cases}$$