Problem 1

$x(t)$ periodic with fundamental period T and fundamental frequency $\omega_0 = 2\pi/T$.

$$x_n = \frac{1}{T}\int_T x(t)\, e^{-jn\omega_0 t}\, dt$$

$$x(t) = \sum_n x_n e^{jn\omega_0 t}$$

(a) $\overline{x}(t) \triangleq x(t - T/2)$ Clearly $\overline{x}(\cdot)$ is also periodic with T and ω_0. Have

$$\overline{x}_n = \frac{1}{T}\int_T \overline{x}(t)\, e^{-jn\omega_0 t}\, dt = \frac{1}{T}\int_0^T x(t-T/2)\, e^{-jn\omega_0 t}\, dt$$

C.O.V. Let $t' = t - T/2$, $dt' = dt$

$$\overline{x}_n = \frac{1}{T}\int_{-T/2}^{T/2} x(t')\, e^{-jn\omega_0 (t'+T/2)}\, dt' = \frac{1}{T}\, e^{-jn\omega_0 T/2}\int_{-T/2}^{T/2} x(t) e^{-jn\omega_0 t}\, dt$$

$\frac{\omega_0 T}{2} = \pi$

$$= e^{-jn\omega_0 T/2}\, x_n = e^{-jn\pi}\, x_n$$

$$\overline{x}_n = (-1)^n x_n$$

(b) $f_n = \begin{cases} x_n & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$

From the above can see that $f_n = \dfrac{x_n + \overline{x}_n}{2}$. From linearity of Fourier series and uniqueness of Fourier series representations we have

$$f(t) = \sum_n f_n e^{jn\omega_0 t} = \frac{x(t) + \overline{x}(t)}{2}$$

$$= \frac{x(t) + x(t - T/2)}{2}$$

Problem 1 (cont'd)

Note that $f(t)$ is periodic with period $T/2$ corresponding to a new fundamental frequency of $\omega_0' = 2\pi/(T/2) = \frac{4\pi}{T} = 2\omega_0$.

Since the odd-indexed Fourier coefficients are equal to zero can also write

$$f(t) = \sum_{n=-\infty}^{\infty} f_n e^{jn\omega_0 t} = \sum_{k=-\infty}^{\infty} f_{2k} e^{j2k\omega_0 t}$$

$$= \sum_{k=-\infty}^{\infty} f_{2k} e^{jk\omega_0 (2t)} = \tilde{f}(2t)$$

ie $\tilde{f}(t) = f(t/2)$ ie the two expressions are related by time scaling.

$\uparrow$ same as $f(\cdot)$ but with time expanded st. $\tilde{f}(\cdot)$ still has fund. period $= T$.

$$\tilde{f}(t) = \frac{x(t/2) + x\left(\frac{t-T}{2}\right)}{2}$$

(c) Using the result of part (a) we have

$$g_n = \frac{x_n - \bar{x}_n}{2}$$

$$\Rightarrow \quad g(t) = \frac{x(t) - \bar{x}(t)}{2} = \frac{x(t) - x(t - T/2)}{2}$$

<u>Note</u>: A typo was made in definition of $\tilde{g}(t)$. See next page.

(c) (cont'd.)

For part ii there was a typo-graphical error. Since the $g_n = 0$ for n odd we must have

$$\tilde{g}(t) = \sum_n g_{2n} e^{jn\omega_0 t} = 0 \quad \forall t \quad \text{since } g_{2n} = 0.$$

This is not what was intended, but it is correct given the typo.

<u>What was intended</u>

$$\tilde{\tilde{g}}(t) = \sum_{n=-\infty}^{\infty} g_{2n+1} e^{jn\omega_0 t}$$

Start with

$$g(t) = \sum_{n=-\infty}^{\infty} g_n e^{jn\omega_0 t} = \sum_{\substack{n=-\infty \\ n \text{ odd}}}^{\infty} g_n e^{jn\omega_0 t}$$

$$= \sum_{k=-\infty}^{\infty} g_{2k+1} e^{j(2k+1)\omega_0 t}$$

$$= e^{j\omega_0 t} \sum_{k=-\infty}^{\infty} g_{2k+1} e^{jk\omega_0 (2t)} = e^{j\omega_0 t} \tilde{\tilde{g}}(2t).$$

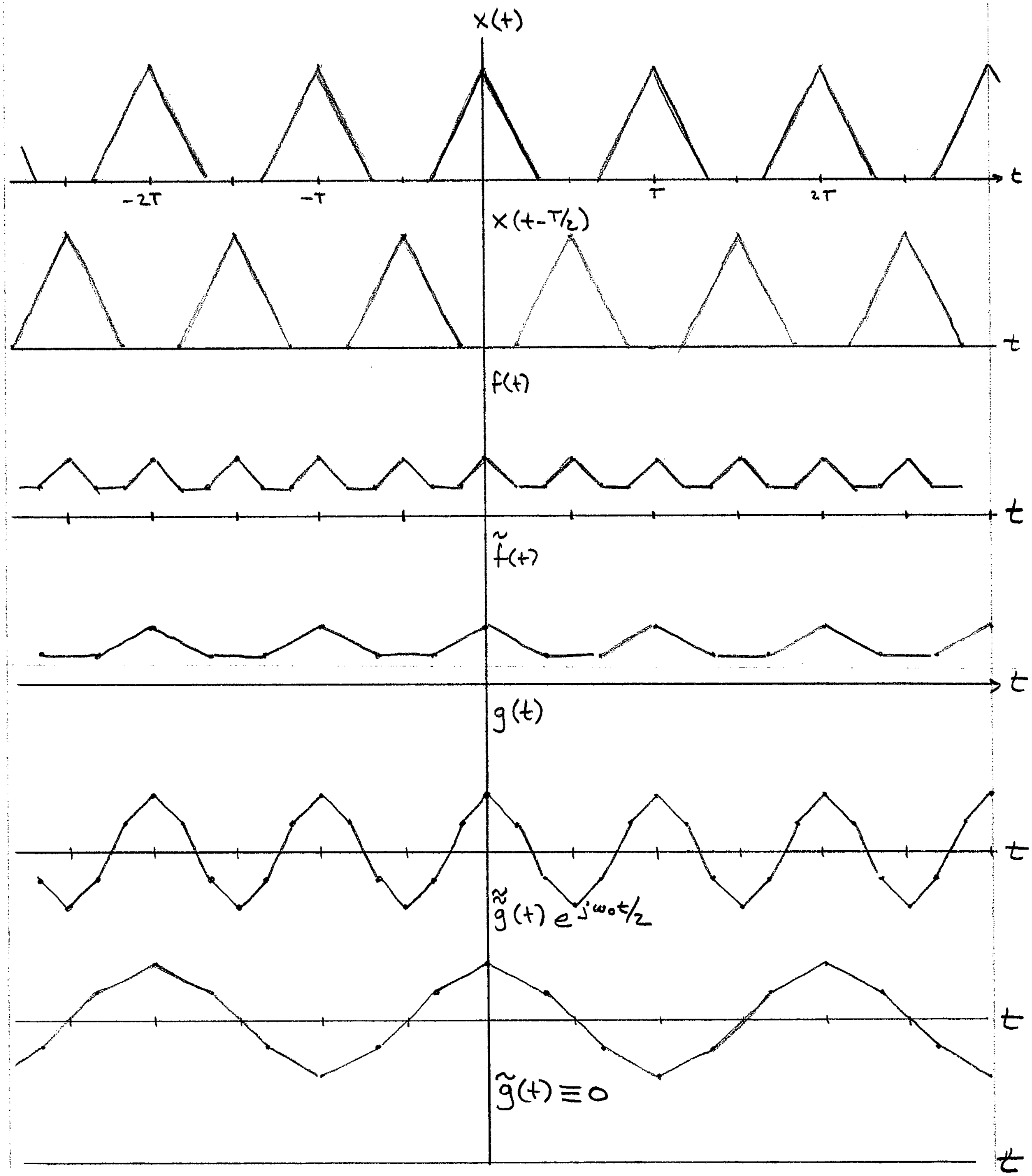
$$\therefore \tilde{\tilde{g}}(2t) = e^{-j\omega_0 t} g(t)$$

$$\tilde{\tilde{g}}(t) = \underbrace{e^{-j\omega_0 t/2}}_{\text{a phase term.}} g(t/2)$$

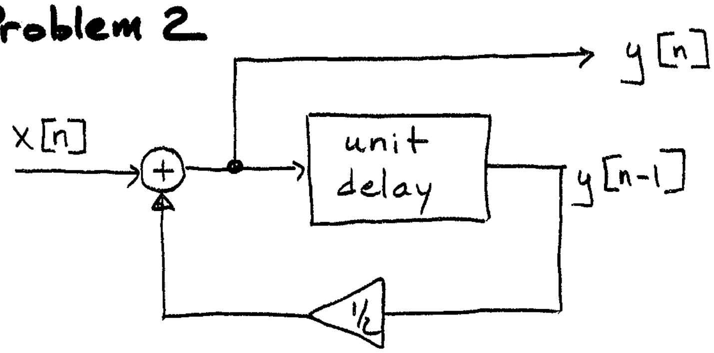
(In the plot on next page we show

$$e^{j\omega_0 t/2} \tilde{\tilde{g}}(t) = g(t/2) = \frac{x(t/2) - x\left(\frac{t-T}{2}\right)}{2} \text{).}$$

Problem 1 (cont'd.)



Problem 2



(a) Find the difference equation

$$y[n] = \frac{1}{2} y[n-1] + x[n]$$

(b) Find the causal impulse response of the above system.

Here replace $x[n] = \delta[n]$ and assume system at rest at $n=-1$ ie $y[-1]=0$. Solution to the equation will be $h[n]$.

$\therefore$ homog. eqn $n>0$ $\quad h[n] - \frac{1}{2} h[n-1] = 0$

The characteristic equation is

$$z - \frac{1}{2} = 0$$

$$\therefore \quad h[n] = K\left(\frac{1}{2}\right)^n \qquad n>0$$

Find the proper initial conditions

$$h[0] = \delta[0] = 1$$

$$h[1] = \frac{1}{2} h[0] = \frac{1}{2}$$

$$\therefore \quad K = 1$$

$$h[n] = \left(\frac{1}{2}\right)^n u[n]$$

Problem 2 (cont'd.)

(c) The impulse response for this new system is clearly

$$h_{new}[n] = a\,h[n] + b\,h[n-1]$$

$$= a\left(\tfrac{1}{2}\right)^n u[n] + b\left(\tfrac{1}{2}\right)^{n-1} u[n-1]$$

(d) $$h_{new}[n] = \begin{cases} 0 & n < 0 \\ a & n = 0 \\ a\left(\tfrac{1}{2}\right)^n + b\left(\tfrac{1}{2}\right)^{n-1} & n \geq 1 \end{cases}$$

Thus need $a = 1$ and

$$a\left(\tfrac{1}{2}\right)^n + b\left(\tfrac{1}{2}\right)^{n-1} = 0 \quad \text{for } n \geq 1$$

This is if and only if

$$\left(\tfrac{1}{2}\right)^n \left\{ a + 2b \right\} = 0 \quad \text{for } n \geq 1.$$

ie $$a + 2b = 0 \implies b = -a/2 = -1/2$$

(e) The inverse system has impulse response

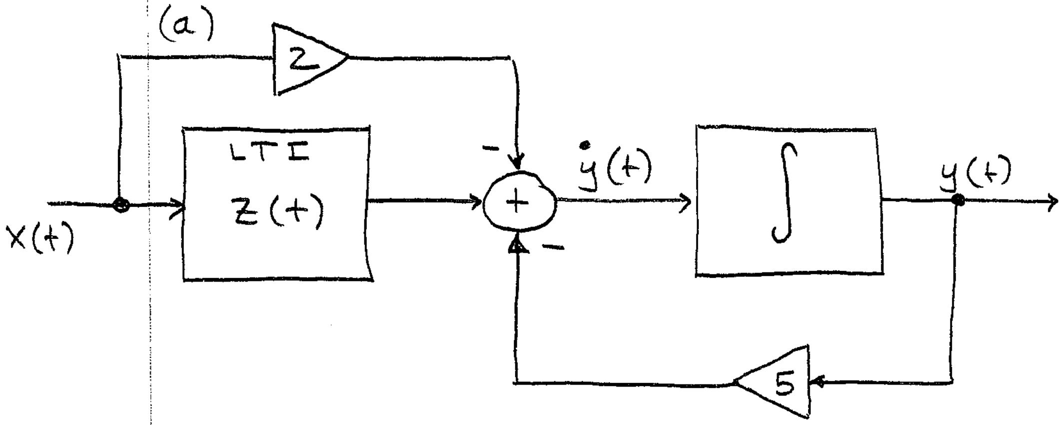
$$h_{inverse}[n] = \begin{cases} 1 & n = 0 \\ -1/2 & n = 1 \\ 0 & \text{otherwise} \end{cases}$$

Problem 3

$$\frac{dy}{dt} + 5y = \int_{-\infty}^{\infty} x(\tau)\, z(t-\tau)\, d\tau - 2x(t)$$

$x(t) =$ input
$y(t) =$ output
$z(t) =$ impulse resp. of system.

(a)



(b) Taking Fourier transform of the defining integro-diff. equation and recognizing the integral as a convolution

$$j\omega Y(j\omega) + 5Y(j\omega) = Z(j\omega)X(j\omega) - 2X(j\omega)$$

$$(j\omega + 5)Y(j\omega) = (Z(j\omega) - 2)X(j\omega)$$

$$\therefore \quad H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{Z(j\omega) - 2}{j\omega + 5}$$

(c) $z(t) = e^{-2t}u(t) + \delta(t)$

(c-i) From transform table and linearity we have

$$Z(j\omega) = \frac{1}{j\omega + 2} + 1$$

(c-ii) Plug into the expression for $H(j\omega)$ and simplify

$$H(j\omega) = \frac{\frac{1}{j\omega+2} - 1}{j\omega + 5} = \frac{1 - j\omega - 2}{(j\omega+5)(j\omega+2)}$$

$$= \frac{-j\omega - 1}{(j\omega+5)(j\omega+2)}$$

Problem 3 (cont'd)

(c-iii) Because $H(j\omega)$ is a rational function in $j\omega$ the overall input-output system is model by an ordinary finite-order differential equation.

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{-j\omega - 1}{(j\omega)^2 + 7(j\omega) + 10}.$$

$$(j\omega)^2 Y(j\omega) + 7j\omega Y(j\omega) + 10Y(j\omega) = -j\omega X(j\omega) - X(j\omega).$$

$$\frac{d^2}{dt^2}y(t) + 7\frac{d}{dt}y(t) + 10\,y(t) = -\frac{d}{dt}x(t) - x(t)$$

(c-iv) with $v = j\omega$

$$H(v) = \frac{-v-1}{(v+5)(v+2)} = \frac{A}{v+5} + \frac{B}{v+2}$$

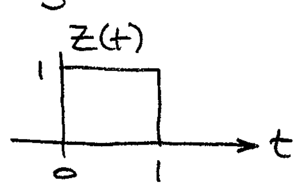
$$A = (v+5)H(v)\Big|_{v=-5} = \frac{5-1}{-3} = -4/3$$

$$B = (v+2)H(v)\Big|_{v=-2} = \frac{1}{3}$$

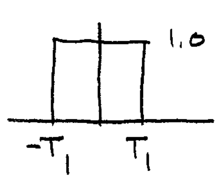
$\therefore$ From Table

$$h(t) = \frac{1}{3}e^{-2t}u(t) - \frac{4}{3}e^{-5t}u(t)$$

(d) Now



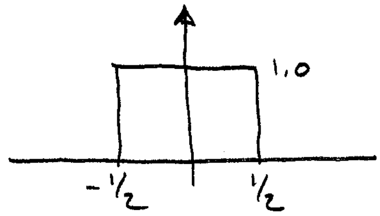
(d-i) From the Fourier transform table



$$\longleftrightarrow \quad \frac{2\sin\omega T_1}{\omega}$$

Problem 3 (cont'd.)

Thus



$$\longleftrightarrow \quad \frac{2 \sin \omega/2}{\omega}$$

But our $z(t)$ is the above time domain signal delayed by $1/2$ sec. From the time-shift property

$$z(t) \longleftrightarrow e^{-j\omega/2}\, 2\frac{\sin \omega/2}{\omega} = Z(j\omega)$$

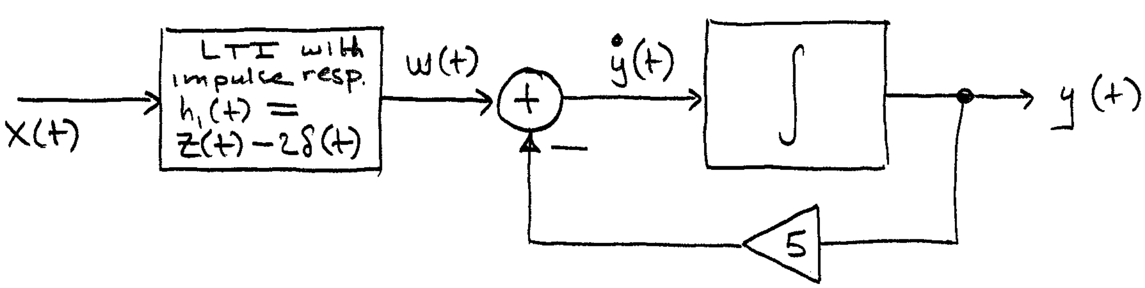
(d-ii) Plugging the expression into the expression for $H(j\omega)$ get

$$H(j\omega) = \frac{\frac{2e^{-j\omega/2}\sin \omega/2}{\omega} - 2}{j\omega + 5}$$

$$= \frac{2e^{-j\omega/2}\sin \omega/2 - 2\omega}{\omega(j\omega+5)}$$

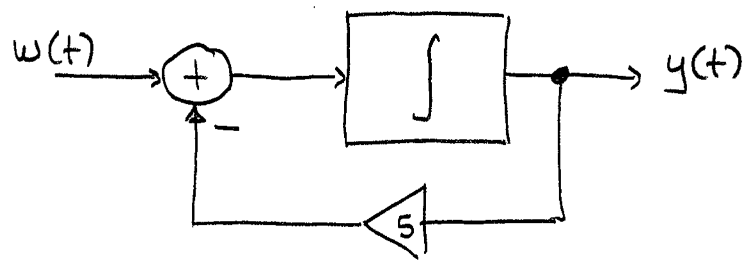
(d-iii) This is <u>not</u> a rational function of $j\omega$ and so the input-output system does <u>not</u> have a finite order differential equation model.

(d-iv) Can't use our usual method to find the overall impulse response. However, a time domain solution can be used in this case. Can redraw the block diagram:



Problem 3 (cont'd.)

Can find the impulce response of the last part



$$\frac{d}{dt}y(t) = -5y(t) + w(t)$$

The impulce response is found to be

$$h_2(t) = e^{-5t}u(t).$$

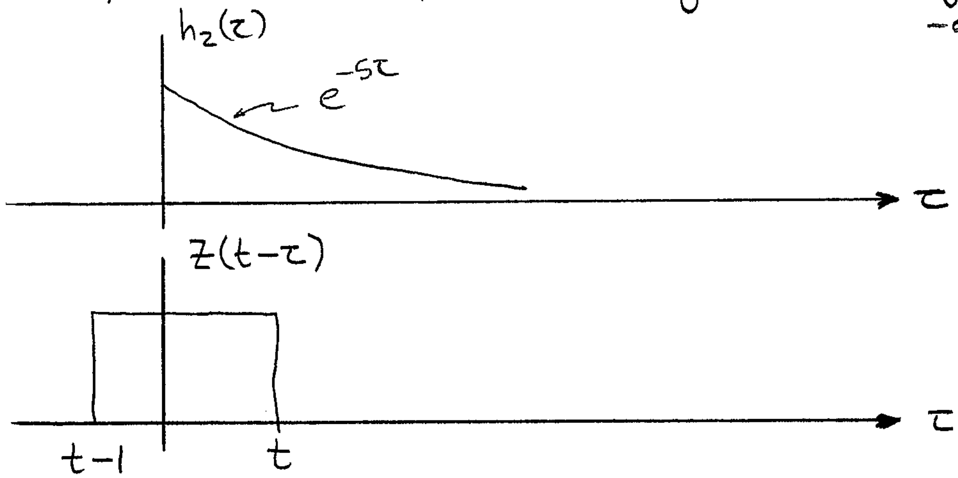
Then the overall impulse response can be found from time domain convolution

$$h(t) = h_1 * h_2(t)$$

$$= (z - 2\delta) * h_2(t)$$

$$= z * h_2(t) - 2h_2(t)$$

This part we compute directly: $z*h_2(t) = \int_{-\infty}^{\infty} z(t-\tau)h_2(\tau)\,d\tau$



Problem 3 (cont'd.)

From the picture there are 3 cases to consider in the computation of $z * h_2(t)$

Case: $t < 0 \implies z * h_2(t) = 0$

Case: $0 < t < 1 \implies$

$$z * h_2(t) = \int_0^t e^{-5\tau}\, d\tau = \frac{1}{-5} e^{-5\tau}\Big|_0^t$$

$$= -\frac{1}{5}\left(e^{-5t} - 1\right) = \frac{1 - e^{-5t}}{5}$$

Case: $t > 1 \implies$

$$z * h_2(t) = \int_{t-1}^t e^{-5\tau}\, d\tau = -\frac{1}{5} e^{-5\tau}\Big|_{t-1}^t$$

$$= -\frac{1}{5}\left(e^{-5t} - e^5 e^{-5t}\right)$$

$$= e^{-5t}\, \frac{e^5 - 1}{5}$$

Putting the pieces back together

$$h(t) = \begin{cases} 0 & t < 0 \\ \frac{1 - e^{-5t}}{5} - 2e^{-5t} & 0 < t < 1 \\ \left(\frac{e^5 - 1}{5}\right) e^{-5t} - 2e^{-5t} & t > 1 \end{cases}$$

$$= \begin{cases} 0 & t < 0 \\ \frac{1 - 11e^{-5t}}{5} & 0 < t < 1 \\ \left(\frac{e^5 - 11}{5}\right) e^{-5t} & t > 1. \end{cases}$$