Prob 4.12

7/15 ①

Use duality to find FT of $x(t) = \frac{2a}{t^2+a^2}$

(a: constant)

$$X(\omega) = \int_{-\infty}^{\infty} \frac{2a}{t^2+a^2} e^{-j\omega t} dt = ?$$

→ hard integral to do.

Instead, employ duality:

since $e^{-a|t|} \overset{\mathcal{F}}{\longleftrightarrow} \frac{2a}{\omega^2+a^2}$

Then $\frac{2a}{t^2+a^2} \overset{\mathcal{F}}{\longleftrightarrow} 2\pi e^{-a|-\omega|} = 2\pi e^{-a|\omega|}$

Using "$X(t) \overset{\mathcal{F}}{\longleftrightarrow} 2\pi x(-\omega)$"

Prob 4.23

$$x_o(t) = \begin{cases} e^{-t}, & 0<t<1 \\ 0, & o/w \end{cases}$$

$$= e^{-t}\{u(t) - u(t-1)\}$$

$$= \underline{e^{-t}u(t)} - e^{-1}\underline{e^{-(t-1)}u(t-1)}$$

Since

$$\left(\begin{array}{l} e^{-at}u(t) \xleftrightarrow{\mathcal{F}} \dfrac{1}{a+j\omega} \\ x(t-t_0) \xleftrightarrow{\mathcal{F}} e^{-j\omega t_0}X(\omega) \end{array} \right.$$

We have:

$$X_o(\omega) = \frac{1}{1+j\omega}\left(1 - e^{-1}e^{-j\omega}\right)$$

$$= \frac{1-e^{-1}(\cos\omega - j\sin\omega)}{1+j\omega} \cdot \frac{1-j\omega}{1-j\omega}$$

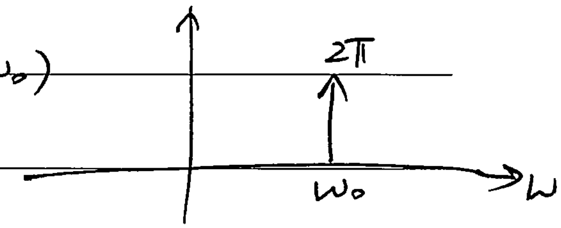
$$= \frac{2-2e^{-1}\cos(\omega) - 2e^{-1}\omega\sin(\omega)}{1+\omega^2} \rightarrow Re\{X_o(\omega)\}$$

$$+ j\left\{\frac{-2\omega + 2e^{-1}\sin(\omega) + 2e^{-1}\omega\cos(\omega)}{1+\omega^2}\right\} \rightarrow Im\{X_o(\omega)\}$$

Fourier Transform of a Finite-Length Sinewave

(Recall)

$$x(t) = e^{j\omega_0 t} \quad (-\infty < t < \infty) \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad X(\omega) = 2\pi\,\delta(\omega - \omega_0)$$



- only freq content is ω_0, so all energy concentrated at $\omega = \omega_0$.
- consider finite-duration sinewave "turned-on" for T secs.

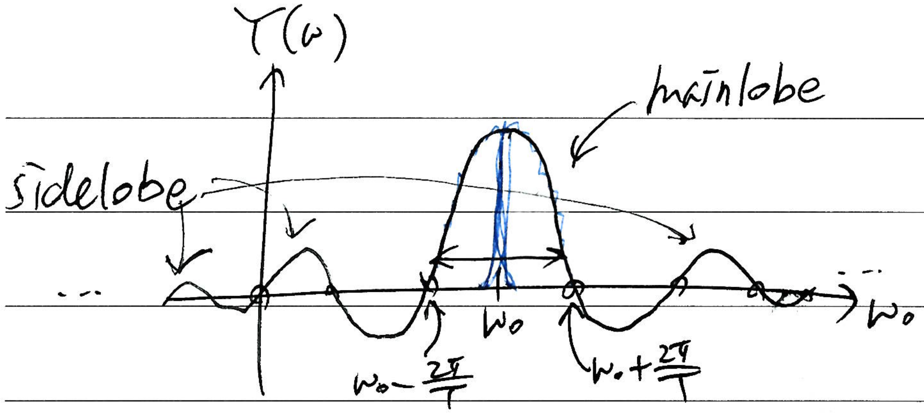
$$y(t) = e^{j\omega_0 t}\,\mathrm{rect}\left(\frac{t}{T}\right) \overset{\mathcal{F}}{\longleftrightarrow} Y(\omega) = ?$$

(Recall)

$$\mathrm{rect}\left(\frac{t}{T}\right) \overset{\mathcal{F}}{\longleftrightarrow} \frac{\sin\left(T\frac{\omega}{2}\right)}{\frac{\omega}{2}}$$

$$x(t)\,e^{j\omega_0 t} \overset{\mathcal{F}}{\longleftrightarrow} X(\omega - \omega_0)$$

$$Y(\omega) = \frac{\sin\left(T\frac{(\omega-\omega_0)}{2}\right)}{\frac{(\omega-\omega_0)}{2}}$$



- zero crossings at $\omega = \omega_0 + \frac{2\pi}{T}\cdot m$, m : Integer $\neq 0$, $-\infty < m < \infty$

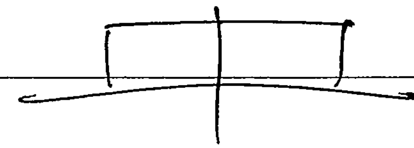
- finite-duration sinewave has energy over a continuum of freq., although something like 88% of the energy is in $\omega_0 - \frac{2\pi}{T} < \omega < \omega_0 + \frac{2\pi}{T}$

- as $T \to \infty$

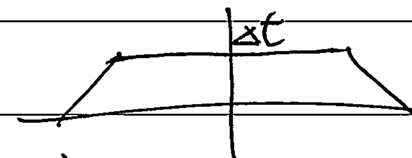
$$\lim_{T\to\infty} e^{j\omega_0 t}\,\mathrm{rect}\left(\frac{t}{T}\right) \xleftrightarrow{\mathcal{F}} \lim_{T\to\infty} \frac{\sin\left(T\frac{(\omega-\omega_0)}{2}\right)}{\frac{(\omega-\omega_0)}{2}}$$

$$= e^{j\omega_0 t} \xleftrightarrow{\mathcal{F}} = 2\pi\,\delta(\omega-\omega_0)$$

Typically, in practice, we taper at both end, i.e, have the sinewave turn on "gradually" rather than instantaneously (turn off "gradually") so that there is more energy in the mainlobe and the sidelobe decay more quickly.



(for example) $rect\left(\frac{t}{T}\right) * rect\left(\frac{t}{\Delta t}\right) \Rightarrow$



$$e^{j\omega_0 t}\left(rect\left(\frac{t}{T}\right) * \frac{1}{\Delta t}\, rect\left(\frac{t}{\Delta t}\right)\right) \stackrel{\mathcal{F}}{\longleftrightarrow} Z(\omega-\omega_0)$$

where $Z(\omega) = \dfrac{\sin\left(T\frac{\omega}{2}\right)}{\frac{\omega}{2}} \cdot \dfrac{1}{\Delta t} \cdot \dfrac{\sin\left(\Delta t\frac{\omega}{2}\right)}{\frac{\omega}{2}}$

→ we have ω^2 in denominator so sidelobes delay more quickly.

Fourier Transform of a periodic signal.

$$x(t) = \sum_{k=-\infty}^{\infty} a_k \underline{e^{jk\frac{2\pi}{T}t}} \overset{\mathcal{F}}{\longleftrightarrow} X(\omega) = \sum_{k=-\infty}^{\infty} a_k \underline{2\pi\,\delta(\omega - k\tfrac{2\pi}{T})}$$

- Alternative method for computing FS coeff. (~ Prob 4.27)

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, e^{-jk\frac{2\pi}{T}t}\, dt$$

$$= \frac{1}{T}\underline{\int_{-\infty}^{\infty} \{\underline{x(t)\,\mathrm{rect}(\tfrac{t}{T})}\}\, e^{-j\omega t}\, dt}\Big|_{\omega = k\frac{2\pi}{T}}$$

$$= \frac{1}{T}\,\mathcal{F}\{\text{one period of } x(t)\}\Big|_{\omega = k\frac{2\pi}{T}}$$

- As a check, we previously determined FS coeff. for periodic train of rectangular pulses.

$$a_k = \frac{\sin(k\pi\frac{\tau}{T})}{k\pi} \quad \text{for} \quad x(t) = \sum_{n=-\infty}^{\infty} \mathrm{rect}\left(\frac{t-nT}{\tau}\right)$$

one period is $\mathrm{rect}\left(\frac{t}{\tau}\right) \overset{\mathcal{F}}{\longleftrightarrow} \dfrac{\sin(\tau\frac{\omega}{2})}{\frac{\omega}{2}}$

$$a_k = \frac{1}{T}\frac{\sin\left(\tau\frac{\omega}{2}\right)}{\frac{\omega}{2}}\Bigg|_{\omega=k\frac{2\pi}{T}} = \frac{1}{\cancel{T}}\frac{\sin\left(\tau\frac{1}{\cancel{2}}k\frac{\cancel{2}\pi}{T}\right)}{\frac{1}{\cancel{2}}k\frac{\cancel{2}\pi}{\cancel{T}}}$$

$$= \frac{\sin\left(k\pi\frac{\tau}{T}\right)}{k\pi}$$ checks!

- suppose signal is only periodic for N periods

$$y(t) = x(t)\,\mathrm{rect}\left(\frac{t}{NT}\right), \quad \text{where } x(t) = x(t+T), \ \forall t$$

Using Multiplication Property of FT,

$$Y(\omega) = \frac{1}{2\pi}X(\omega) * \frac{\sin\left(NT\frac{\omega}{2}\right)}{\frac{\omega}{2}}$$

$$= \frac{1}{\cancel{2\pi}}\sum_{k=-\infty}^{\infty} a_k \cancel{2\pi}\,\delta\left(\omega - k\frac{2\pi}{T}\right) * \frac{\sin\left(NT\frac{\omega}{2}\right)}{\frac{\omega}{2}}$$

$$= \sum_{k=-\infty}^{\infty} a_k \frac{\sin\left(NT\frac{\left(\omega-k\frac{2\pi}{T}\right)}{2}\right)}{\frac{\left(\omega-k\frac{2\pi}{T}\right)}{2}}$$

Passing sinewaves thru LTI system

$$x(t) = e^{j\omega_0 t} \quad \forall t \longrightarrow \boxed{\begin{array}{c}\text{LTI} \\ h(t) \leftrightarrow H(\omega)\end{array}} \longrightarrow y(t) = H(\omega_0)\, e^{j\omega_0 t}$$

Recall: this is where the defining integral eq. for the FT came from

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} h(\tau)\, \underbrace{e^{j\omega_0 (t-\tau)}}_{= x(t-\tau)}\, d\tau$$

$$= e^{j\omega_0 t} \int_{-\infty}^{\infty} h(\tau)\, e^{-j\omega_0 \tau}\, d\tau$$

$$= H(\omega_0)\, e^{j\omega_0 t}, \quad \text{where } H(\omega) = \int_{-\infty}^{\infty} h(t)\, e^{-j\omega t}\, dt$$

Thus: $\boxed{e^{j\omega_0 t} * h(t) = H(\omega_0)\, e^{j\omega_0 t}}$

→ no reason to do convolution.