SD

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(15 pts) **1.** Using the definition of the Fourier transform (*not* the table of Fourier transform pairs), compute the Fourier transform of the DT signal:

$$x[n] = \left(\frac{1}{2j}\right)^{|n|}.$$

$\left(\frac{1}{2j}\right)^{|n|} = \begin{cases} \left(\frac{1}{2j}\right)^{n} ; & n \geq 0 \\ \left(\frac{1}{2j}\right)^{-n} ; & n < 0 \end{cases}$

$= \left(\frac{1}{2j}\right)^{n} u[n] + \left(\frac{1}{2j}\right)^{-n} u[-n+1]$

$X(\omega) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$

$= \sum_{n=-\infty}^{\infty} \left(\frac{1}{2j}\right)^{|n|} e^{-j\omega n}$

$= \sum_{n=-\infty}^{0} \left(\frac{1}{2j}\right)^{-n} e^{-j\omega n} + \sum_{n=0}^{\infty} \left(\frac{1}{2j}\right)^{n} e^{-j\omega n}$

$= \sum_{n=-\infty}^{0}$

$= \sum_{n=-\infty}^{-1} \left(\frac{1}{2j}\right)^{-n} e^{-j\omega n} + \sum_{n=0}^{\infty} \left(\frac{1}{2j}\right)^{n} e^{-j\omega n}$

$= \sum_{n=-\infty}^{-1} \left(\frac{e^{j\omega}}{2j}\right)^{-n} + \sum_{n=0}^{\infty} \left(\frac{1}{2j e^{j\omega}}\right)^{n}$

let $k=-n$

$= \sum_{k=1}^{\infty} \left(\frac{e^{j\omega}}{2j}\right)^{k} + \sum_{n=0}^{\infty} \left(\frac{1}{2j e^{j\omega}}\right)^{n}$

$= \frac{1}{1-\frac{e^{j\omega}}{2j}} - 1 + \frac{1}{1-\frac{1}{2je^{j\omega}}}$

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(20 pts) **2.** The Frequency response of a continuous-time LTI system is

$$H(j\omega) = \mathcal{H}(\omega) = \frac{1}{j\omega + 2}.$$

Use the convolution property of the Fourier transform to determine the response $y(t)$ when the input is $x(t) = e^{-|t|}$.

$H(\omega) = \frac{1}{j\omega+2}$, $x(t) = e^{-t}$; $t > 0$
$x(t) = e^{t}$; $t < 0$

$x(t) = e^{-t} u(t) + e^{t} u(-t)$

$X(\omega) = \mathcal{F}(x(t)) = \mathcal{F}(e^{-t} u(t)) + \mathcal{F}(e^{t} u(-t))$ [by (9)]

$= \frac{1}{1+j\omega} + \frac{1}{1-j\omega}$ [by (7) of time reversal]

by convolution property of F.T.
$H(\omega) = \frac{Y(\omega)}{X(\omega)} \Rightarrow Y(\omega) = H(\omega) X(\omega)$

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$= \frac{1}{j\omega+2} \cdot \left[\left(\frac{1}{j\omega+1}\right) + \left(\frac{1}{1-j\omega}\right)\right]$

$= \frac{1}{(j\omega+1)(j\omega+2)} + \frac{1}{(j\omega+2)(1-j\omega)}$

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$= \frac{1}{j\omega+1} - \frac{1}{j\omega+2} + \frac{1}{3}\left(\frac{1}{j\omega+2}\right) - \frac{1}{3}\left(\frac{1}{j\omega-1}\right)$

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$Y(\omega) = \frac{1}{j\omega+1} - \frac{2}{3} \cdot \frac{1}{(j\omega+2)} - \frac{1}{3} \cdot \frac{1}{(j\omega-1)}$

$y(t) = \mathcal{F}^{-1}(Y(\omega)) = e^{-t} u(t) - \frac{2}{3} e^{-2t} u(t) - \frac{1}{3} e^{t} u(t)$ -2

$+ 1/3\, e^{t} u(-t)$.

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(15 pts) **3.** True/False? The Fourier transform of a DT signal $x[n]$ is a periodic function, no matter what $x[n]$ is. (Justify your answer.)

True

justification

let $x[n]$ be any D.T. signal.

$$\mathcal{F}(x[n]) = X(\omega) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

$$X(\omega + 2\pi \cancel{l}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}\, \underbrace{e^{-j2\pi l n}}_{1}$$

$$= \sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n}$$

$$= X(\omega)$$

hence $X(\omega)$ is periodic with periodic 2π no matter what $x[n]$ is.

The F.T. of a D T signal is periodic with period 2π because it is a linear combination of function which are periodic with period 2π

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C.T.

(10 pts) **4.** A continuous-time LTI system has frequency response

$$H(j\omega) = \frac{j\omega + 4}{(j\omega + 2)(j\omega + 3)}.$$

Derive a differential equation representing this system. (Use the properties of the Fourier transform listed in the table to justify your answer.)

$$H(j\omega) = H(\omega) = \frac{j\omega+4}{(j\omega+2)(j\omega+3)}.$$

$$H(\omega) = \frac{Y(\omega)}{X(\omega)} = \frac{j\omega+4}{(j\omega+2)(j\omega+3)}$$

$$Y(\omega)[(j\omega+2)(j\omega+3)] = (j\omega+4)\,X(\omega).$$

$$Y(\omega)\{j^2\omega^2+3j\omega+2j\omega+6\} = X(\omega)j\omega + 4X(\omega)$$

$$Y(\omega)[-\omega^2+5j\omega+6] = j\omega X(\omega) + 4X(\omega).$$

$$\mathcal{F}^{-1}(-\omega^2 Y(\omega) + 5j\omega Y(\omega) + 6Y(\omega)) \; \mathcal{F}^{-1}(j\omega X(\omega) + 4X(\omega))$$

$$-\omega^2 y(t) + 5\frac{d}{dt}y(t) + 6y(t) = \frac{d}{dt}x(t) + 4x(t).$$

$$\boxed{(6-\omega^2)y(t) + 5\frac{d}{dt}y(t) = 4x(t) + \frac{d}{dt}x(t).}$$ (by 9,16)

$$(6-\omega^2)Y(\omega) + 5j\omega Y(\omega) = 4X(\omega) + j\omega X(\omega)$$

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$$\frac{Y(\omega)}{X(\omega)} = \frac{4+j\omega}{(6-\omega^2+5j\omega)} = \frac{4+j\omega}{(j\omega+2)(j\omega+3)}$$

$+5j\omega+6$

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(10 pts) **5.** A CT signal $x(t)$ has Fourier transform

$$\mathcal{X}(\omega) = -2e^{(j-1)\omega}u(\omega+1).$$

$x(t-6)$

Denote by $y(t)$ the signal obtained by delaying $x(t)$ by six seconds. Sketch a graph representing the magnitude $|\mathcal{Y}(\omega)|$ of the Fourier transform $\mathcal{Y}(\omega)$ of $y(t)$. (Justify your answer.)

$$y(t) = x(t-6).$$

$$Y(\omega) = \mathcal{F}(y(t)) = \mathcal{F}(x(t-6)) = e^{-j\omega(6)}X(\omega). \checkmark \text{ (by (10))}$$

$$e^{j\omega - \omega \cdot 6j}$$

$$= -2e^{(j-1)\omega}u(\omega+1)\,e^{-6j\omega}.$$

$$= -2u(\omega+1)\,e^{j\omega}e^{-\omega}e^{-6j\omega}$$

$$= -2u(\omega+1)\,e^{-5j\omega}e^{-\omega}.$$

$$= -2e^{-(1+5j)\omega}u(\omega+1)$$

$$= \begin{cases} -2e^{-(1+5j)\omega}, & \text{when } \omega \geq -1. \\ 0 & \text{when } \omega \leq -1 \end{cases}$$

$$|Y(\omega)| = \begin{cases} 2|e^{-\omega}| = 2e^{-\omega}; & \text{when } \omega > -1 \\ 0; & \text{when } \omega \leq -1 \end{cases}$$