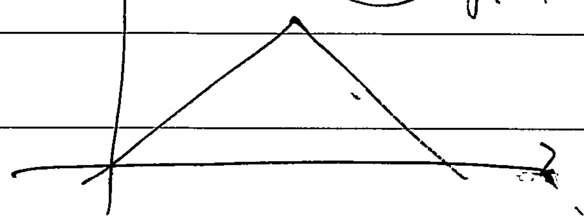
7/9 ①

- Closed-form expression. : math. expression that can be evaluated in a finite number of operations

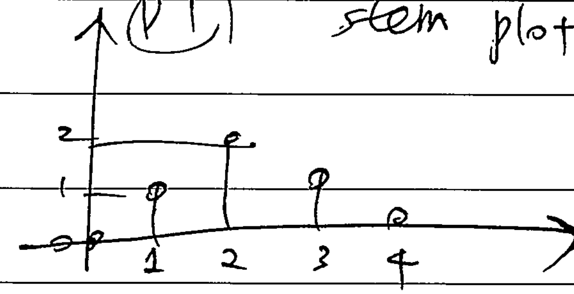
  e.g. $y(t) = -2x(t-2) + -1x(t-1) + x(t) + \cdots$

  $y(t) = -2x(t-2) + x(t)^3$

- CT plot



- DT stem plot



- sequence $\Rightarrow \{\underset{\substack{\uparrow \\ n=0}}{0}, 1, 2, 1, 0\}$

Fourier Transform Properties (Proofs and Examples)

1) Time-shift Property.

$$x(t) \xleftrightarrow{\mathcal{F}} X(\omega)$$

Inverse FT

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$$

$$x(t-t_0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega(t-t_0)}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \left(e^{-j\omega t_0} X(\omega)\right) e^{j\omega t}\, d\omega$$

Since FT is unique,

$$\boxed{x(t-t_0) \xleftrightarrow{\mathcal{F}} e^{-j\omega t_0} X(\omega)}$$

(freq.-shift)

2) Modulation Property

FT : $X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$

$$X(\omega-\omega_0) = \int_{-\infty}^{\infty} x(t)\, e^{-j(\omega-\omega_0)t}\, dt$$

$$= \int_{-\infty}^{\infty} \left\{e^{j\omega_0 t} x(t)\right\} e^{-j\omega t}\, dt$$

$$\boxed{e^{j\omega_0 t} x(t) \xleftrightarrow{\mathcal{F}} X(\omega-\omega_0)}$$

3) Convolution Property.

$$y(t) = x(t) * h(t)$$

Take FT of both sides: (integrate w.r.t τ.)

$$Y(\omega) = \mathcal{F}\{x(t) * h(t)\} = \mathcal{F}\left\{\int_{-\infty}^{\infty} \underline{h(\tau)}\, x(t-\tau)\, d\tau\right\}$$

$$= \int_{-\infty}^{\infty} h(\tau)\, \mathcal{F}\{x(t-\tau)\}\, d\tau$$

time-shift property.

$$= \int_{-\infty}^{\infty} h(\tau)\, e^{-j\omega\tau} X(\omega)\, d\tau$$

$$= X(\omega) \int_{-\infty}^{\infty} h(\tau)\, e^{-j\omega\tau}\, d\tau$$

$$= X(\omega)\, H(\omega)$$

$$\boxed{x(t) * h(t) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega) H(\omega)}$$

4) Multiplication Property.

$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega)$, $y(t) \overset{\mathcal{F}}{\longleftrightarrow} Y(\omega)$

$$z(t) = x(t)\, y(t) \overset{\mathcal{F}}{\longleftrightarrow} Z(\omega) = ?$$

$$Z(\omega) = \int_{-\infty}^{\infty} \{x(t)\, \underline{\underline{y(t)}}\}\, e^{-j\omega t}\, dt$$

substitue inverse FT for $y(t)$; let me use another variable λ

$$y(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} Y(\omega)\, e^{j\omega t}\, d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} Y(\lambda)\, e^{j\lambda t}\, d\lambda$$

$$Z(\omega) = \int_{-\infty}^{\infty} x(t)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} Y(\lambda)\, e^{j\lambda t}\, e^{-j\omega t}\right] d\lambda\, dt$$

Switch order of Integration,

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} Y(\lambda) \underbrace{\int_{-\infty}^{\infty} x(t)\, e^{-j(\omega-\lambda)t}\, dt}_{= X(\omega-\lambda)} d\lambda$$

: modulation property.

$$= \frac{1}{2\pi} \underline{\int_{-\infty}^{\infty} Y(\lambda)\, X(\omega-\lambda)\, d\lambda}$$

$$= \frac{1}{2\pi} X(\omega) * Y(\omega)$$

☆ $$\boxed{x(t)\, y(t) \xleftrightarrow{\mathcal{F}} \frac{1}{2\pi} X(\omega) * Y(\omega)}$$

5) Differentiation Property $\quad \frac{d}{dt}x(t) \overset{\mathcal{F}}{\longleftrightarrow} ?$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}\,d\omega$$

Take derivative wrt t on both sides,

$$\underline{\frac{d}{dt}x(t)} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\frac{d}{dt}\left(e^{j\omega t}\right)d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \underline{\{X(\omega)\, j\omega\}}\, e^{j\omega t}\,d\omega$$

Thus: $\boxed{\frac{d\,x(t)}{dt} \underset{\mathcal{F}}{\longleftrightarrow} j\omega X(\omega)}$

6) Multiplication by Time Property. $\quad t\,x(t) \overset{\mathcal{F}}{\longleftrightarrow} ?$

$$\underline{\frac{d}{d\omega}\{X(\omega)\}} = \frac{d}{d\omega}\left\{\int_{-\infty}^{\infty} x(t)\,e^{-j\omega t}\,dt\right\}$$

$$= \int_{-\infty}^{\infty} x(t)\,\frac{d}{d\omega}\left(e^{-j\omega t}\right)dt$$

$$= \int_{-\infty}^{\infty} \underline{\{x(t)\,(-jt)\}}\, e^{-j\omega t}\,dt$$

Thus: $\boxed{\begin{aligned} -jt\,x(t) &\underset{\mathcal{F}}{\longleftrightarrow} \frac{d}{d\omega}X(\omega) \\ t\,x(t) &\underset{\mathcal{F}}{\longleftrightarrow} j\frac{d}{d\omega}X(\omega) \end{aligned}}$: Differentiation in freq.

7) Duality Property (a little tricky)

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt.$$

$$= \int_{-\infty}^{\infty} x(\lambda)\, e^{-j\omega\lambda}\, d\lambda$$

change to $\lambda = t$

Replace / substitute $\omega = t$ in eqn.

$$X(t) = \int_{-\infty}^{\infty} x(\lambda)\, e^{-jt\lambda}\, d\lambda = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi x(\lambda)\, e^{-j\lambda t}\, d\lambda$$

change of variables: $\begin{pmatrix} \omega = -\lambda \\ d\omega = -d\lambda \end{pmatrix}$

$$\underline{X(t)} = \frac{1}{2\pi}\int_{+\infty}^{-\infty} 2\pi x(-\omega)\, e^{j\omega t}\, (-d\omega)$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \{\underline{2\pi x(-\omega)}\}\, e^{j\omega t}\, d\omega$$

Thus, if $x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(\omega)$

then $X(t) \underset{\mathcal{F}}{\longleftrightarrow} 2\pi x(-\omega)$

8) Time-Scaling Property.

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

$$x(at) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega) e^{j\omega at} d\omega$$

change of variable : $\begin{cases} \omega' = \omega a & \omega = \frac{\omega'}{a} \\ d\omega' = a\, d\omega & d\omega = \frac{d\omega'}{a} \end{cases}$

(assume $a > 0$) $\quad \omega \Big]_{-\infty}^{\infty} \rightarrow \omega' = a\omega \Big]_{-\infty}^{\infty}$

$$x(at) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X\left(\frac{\omega'}{a}\right) e^{j\omega' t} \frac{d\omega'}{a}$$

(drop primes) $= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{a} X\left(\frac{\omega}{a}\right) e^{j\omega t} d\omega$

Thus : $x(at) \longleftrightarrow \frac{1}{a} X\left(\frac{\omega}{a}\right)$

can further prove for any a :

$$\boxed{x(at) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{|a|} X\left(\frac{\omega}{a}\right)}$$

— Summary

Dual Property Pair 1:

$$x(t-t_0) \xleftrightarrow{\mathcal{F}} e^{-j\omega t_0} X(\omega)$$

$$e^{j\omega_0 t} x(t) \xleftrightarrow{\mathcal{F}} X(\omega-\omega_0)$$

Dual Property Pair 2:

$$x(t) * y(t) \xleftrightarrow{\mathcal{F}} X(\omega) Y(\omega)$$

$$x(t) y(t) \xleftrightarrow{\mathcal{F}} \frac{1}{2\pi} X(\omega) * Y(\omega)$$

Dual Property Pair 3:

$$\frac{d}{dt} x(t) \xleftrightarrow{\mathcal{F}} j\omega X(\omega)$$

$$t\, x(t) \xleftrightarrow{\mathcal{F}} j \frac{d}{d\omega} X(\omega)$$

Duality Property. if $x(t) \xleftrightarrow{\mathcal{F}} X(\omega)$ then

$$X(t) \xleftrightarrow{\mathcal{F}} 2\pi x(-\omega)$$

Time-scaling property

$$x(at) \xleftrightarrow{\mathcal{F}} \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$$

Parseval's Theorem

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega \quad : \text{energy}$$

Midterm