LAB #11

LINEAR SYSTEMS

Goal: Investigate the behavior of a linear system of equations near its equilibrium point. Characterize the behavior in terms of the nature of the eigenvalues.

Required tools: MATLAB routine ***pplane*** , `fplot`; solutions of linear homogeneous systems using eigenvalues and eigenvectors.

DISCUSSION

In this lab you will perform a systematic study of the solutions to the linear homogeneous system of differential equations with constant coefficients given by

$$\vec{\mathbf{x}}\,'(t) = A\,\vec{\mathbf{x}}(t) \qquad (*)$$

near its equilibrium point, where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. If we let $\boldsymbol{T = a + d}$ denote the **trace** of A and let $\boldsymbol{D = ad - bc}$ denote the **determinant** of A, then we observe that the characteristic equation of of the matrix A is

$$\lambda^2 - T\lambda + D = 0\,.$$

ASSIGNMENT

(1) Find the equilibrium point(s) for the system $(*)$.

(2) Show that the eigenvalues of the 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ are given by

$$\lambda = \frac{T \pm \sqrt{T^2 - 4D}}{2}\,,$$

where $T = a + d$ and $D = ad - bc$. The eigenvalues will be imaginary if $\dfrac{T^2}{4} < D$. Using `fplot`, plot the function $y = \dfrac{x^2}{4}$ for $|x| \leq 4$ and $|y| \leq 4$. Print your plot and label the axes T and D. (You can get MATLAB to do the labeling. See the help for the routine "plot.")

(a) Indicate on your picture the set of points (T, D) for which the eigenvalues for the corresponding system are not real. Call this region **NR** (for "not real".)

(b) Values of T and D outside **NR** yield real eigenvalues. Indicate on your graph the regions where:

(i) both eigenvalues are positive (Call this region **PP**)

(ii) both are negative (Call this region **NN**)

(iii) one eigenvalue is positive and one is negative. (Call this region **PN**)

(Hint: Your answers will depend in which quadrant of the TD-plane the point (T, D) lies. In the first quadrant, $D > 0$ and $T > 0$ and hence $\sqrt{T^2 - 4D} < T$. What does this say about the sign of $T \pm \sqrt{T^2 - 4D}$? Do a similar analysis for the other quadrants.)

(3) To study a corresponding linear system $\vec{\mathbf{x}}' = A\vec{\mathbf{x}}$ representing various values of T and D, we can let

$$A = \begin{pmatrix} 0 & -1 \\ D & T \end{pmatrix} \qquad (**)$$

because $\operatorname{trace}(A) = T$ and $\det(A) = D$.

Pick an arbitrary point (T, D) in region **PP** (your choice) and enter the system

$$\begin{pmatrix} x1 \\ x2 \end{pmatrix}' = \begin{pmatrix} 0 & -1 \\ D & T \end{pmatrix} \begin{pmatrix} x1 \\ x2 \end{pmatrix} \qquad (***)$$

into ***pplane*** . Plot several orbits and describe their behavior. Specifically, consider the following questions:

(a) Is the origin a *source*, a *sink*, a *center*, or a *saddle* ?
(Recall that an equilibrium point is a
Sink if all solutions sufficiently close to it converge to it;
Source if all solutions sufficiently close to it move away from it;
Center if all solutions sufficiently close to it loop around it;
Saddle if all solutions sufficiently close to it, some converge to it and some move away from it.)

(b) If the origin is a source, do the solutions spiral out as they move away from it ? If the origin is a sink, do the solutions spiral in as they move toward it ? Indicate the point you used on your TD- plane and draw, in a small box near your point (T, D), a small sketch of the phase plane portrait for the corresponding system.

(4) Find the general solution to the system $(***)$, with the same values of T and D you chose in (3).

Note: You can get MATLAB to do all of the work. For example, to find the eigenvalues and eigenvectors for the matrix $A = \begin{pmatrix} -4 & 6 \\ -3 & 5 \end{pmatrix}$, enter the following into MATLAB :

```
>> A=[-4,6;-3,5];
>> [B,D]=eig(A)
```

The diagonal entries of D are the corresponding eigenvalues of A and the columns of B are the corresponding eigenvectors.

(5) Use your answer to the previous exercise to prove that your answers to part (3)(a) would be valid for any solution (with (T, D) in the region **PP**) to the system studied in part (3).

(6) Now choose a point (T, D) in the region **NN** and repeat parts (3) and (4).

(7) Now choose a point (T, D) in the region **NP** and repeat parts (3) and (4). It should turn out that the origin is a saddle. To prove this, you must consider the straight line solutions.

(8) The behavior in region **NR** is somewhat varied. Find examples of points (T, D) for which the corresponding system $(*)$ has

(a) a spiral sink
(b) a spiral source and
(c) a center
at the origin $(0, 0)$.

Prove that your answers in (a), (b) and (c) are correct by finding the general solution to the system $(***)$ for your choices of (T, D). (MATLAB can compute also complex eigenvalues.) Indicate on your picture from part (a) which points in region **NR** would produce which kind of behavior.

For your summary, discuss what kinds of behavior can be expected for the orbits of the system corresponding to equation $(*)$ depending upon in which region (T, D) lies. Relate this behavior to the nature of the eigenvalues of the matrix which describes the system. On the basis of the above work, describe what you feel the nature of the orbits would be in terms of the nature of the eigenvalues of A. For example, if both eigenvalues are negative, is the origin a source, sink , center or saddle. Why ? If the eigenvalues are complex, under what conditions would the origin be a spiral sink and under what conditions would it be a spiral source ? Under what conditions would it be a saddle? A center?